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Torben Klarl

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On the stability of equilibria in replicator dynamics modelling: an application in industrial dynamics considering resource constraints

Torben Klarl
Department of Economics, University of Augsburg,
Universitätsstraße 16, D-86159 Augsburg, Germany
e-mail: torben.alexander.klarl@wiwi.uni-augsburg.de
Tel.: ++49(0)821-598-4177
Fax: ++49(0)821-598-4229

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Abstract

Replicator dynamic modelling (rdm) is used to discuss industrial evolution problems with heterogeneous agents. However, some of the models tend to be very complex and, therefore, analytical solutions cannot be obtained. Hence, the paper proposes to start with a relatively simple model and check its stability of the equilibria before expanding the model. This strategy is more effective than relying on simulation based studies where instability cannot be ruled out ex ante. Thus, the aim of this paper is to introduce a stability check for rdm, especially, if one ore more real Eigenvalues with value zero occur. Besides the (Strogatz, 1994) and (Hilborn, 1994) local stability theorem, this method provides an alternative and more flexible procedure for stability analysis for rdm. To apply this approach, an industrial replicator dynamic model containing three differential equations is set up.

Keywords: Evolutionary economics, Replicator dynamics
JEL Classification Number: B25, C62

1 Motivation

One of the most used techniques to study evolutionary based models is the system dynamics approach. Within this approach it is possible to study the evolution of complex system over time. Usually, this conception make use of so called positive or negative feedback loops, stocks and delays regarding time which sway the behaviour of the system. Due to the fact that the system dynamics approach is based on repeating behavioral patterns, it is possible to simulate different socio economic scenarios, such as the introduction of a specific tax in a macro model or the launch of a specific product in a corresponding diffusion model, and to compare the resulting effects of the entire system.

The replicator dynamics approach is not only closely related, but is an application of the system dynamics approach. (Fisher, 1930) made use of Charles Darwin‘s "survival of the fittest" thesis and formulated the so called replicator equations. Originally, the replicator dynamics approach was used in research fields such as biology and ecology. But today, this tool is widely spread in economic fields and amends some classic disciplines within the economic research agenda. An example is the evolutionary game theory which is based on this approach.

The charm of using the replicator dynamics is, on the one hand, a conceptual one: Evolutionary economics is mainly based on the assumption of heterogeneity concerning agents. Heterogeneous ideas or more precisely the realization of ideas in strategies are of major interest to answer the question about the fitness of strategies in a given strategy pool. The implication is that superior and inferior strategies exist and superior strategies eliminate inferior strategies. On the other hand, it is a computational one because of the improvement of the computer capacities, which allow to evaluate highly complex systems computationally.

A mass of papers has been published using replicator dynamics in various fields. (Cantner and Hanusch, 1998), for instance, built a simple market model to simulate industrial dynamics, (Noailly et al., 2003) exploited this approach to deduce evolutionary harvesting strategies. Some of the models tend to be very complex and, therefore, analytical solutions cannot be obtained. For that reason, one should first start with a relative simple model and check its stability. Afterwards, the model can be expanded in several ways. This strategy is more effective than only relying on simulation based results where instability cannot be ruled out ex ante. With the exception of (Noailly et al., 2003), who give a explicitly proof for a two dimensional system, or (Noailly, 2008), it is not common to check robustness of derived equilibria within the replicator dynamics approach in the relevant literature.

And so, the aim of this paper is to introduce a robustness check for replicator dynamic modelling, especially for the case if more than one Eigenvalue with value zero occur. If two or more real Eigenvalues with value zero appear and all other resulting Eigenvalues are negative, then it is a priori not possible to say if this system is stable or not. (Strogatz, 1994) and (Hilborn, 1994) have pointed out for a system, containing three differential equations, that a steady state is locally asymptotically stable if all resulting eigenvalues of the Jacobian have negative real parts, or two of the three Eigenvalues are negative and one has value zero. Obviously, for this case, the local stability criterion theorem defined by (Strogatz, 1994) and (Hilborn, 1994) cannot be applied.

It is straightforward to see, that the probability of the occurrence of two ore more real Eigenvalues rises with the dimension of the replicator dynamics system. Thus, the stability analysis based on the Eigenvalue criterion is performed not for the common and widely discussed two dimensional case, but for the more complex three dimensional case.

To apply this approach, we set up an industrial dynamic model with an endogenous production process. The production depends on a regenerative but exhaustible resource. Additionally, we integrate technological progress, which reduces the cost of production. Both, the evolution of the resource and the integrated technological progress influence the evolution of the market share. As a result, we obtain a model that consists of three differential equations which has to be solved. Besides the contribution of (Noailly, 2008)‘s recent paper, this paper is one of the very few considering more than two dimensions. In addition, in the model we distinguish between constant, increasing and decreasing returns to scale, and so we create a direct link to "Schumpeterian I"[1] and "Schumpeterian II"[2] hypothesis. Moreover, this model integrates some ideas of (Cantner, 1997), (Cantner and Hanusch, 1998) and (Noailly et al., 2003).

The paper is structured as follows: In the first subsection, we formulate the production sector, where we mainly focus on the production process, which is constraint by using a regenerative, but exhaustible resource as input factor. The next subsection deals with the evolution of the market share. Subsection 2.3. introduces agent specific heterogeneity within the cost reducing process, which is accelerated by technological progress. After setting up the model, we discuss the dynamic behaviour (subsection 2.5.) followed by an extended stability analysis of the resulting equilibria (subsection 2.6.). The paper closes with Section 3, which gives a short summary of the derived results.

2 Resource constraints and industrial dynamics

In this section, we introduce a simple model of the competition of selection, which is widely used in the evolutionary context. This model is based on the seminal work of (Dosi, 1982), who introduced the so called paradigm-trajectory-approach and hereby identified stylized facts for the evolution of an industry.

First, we formulate and introduce the production sector, which is, for example, missing in the work of (Cantner and Hanusch, 1998). Afterwards, we make some comments concerning the modelling of the market structure evolution followed by a motivation of technological progress, which is not included in the work of (Noailly et al., 2003).

2.1 Production sector

In the model economy, we make the assumption that a set of agents or firms $\mathcal{B}$ exists, who produce under usage of n strategies with $i = \{1, 2, ..., h-1, h, h+1, ..., j-$

[1]For instance refer to (Acs and Audretsch, 1987), (Malerba and Orsenigo, 1993) and (Malerba et al., 1997).

[2]Refer to (Cantner and Hanusch, 1998) or (Malerba and Orsenigo, 1993).

$1, j, j+1, ..., n-1, n\}$. To keep the model simple, we further assume that every agent with strategy i produces with a linear production technique with the only input N_t. Hereby N_t can be considered as a regenerative but exhaustable resource[3] which is growing with a certain exogenous and time independent degree ξ in every period t, $t \in T$.

In contrast to the old neoclassical theory, we do not assume a production technique which uses input factors available ad infinitum in extremum. It is more realistic that the production decision depends on scarce resources, especially in the short run.

The reason for this assumption is to endogenize the production decision which is directly linked to the resource dynamics via the cost of production. We refer to this point later on.

In this way, we follow (Noailly et al., 2003), with the exception that we give a wider definition of the evolution of the input factor N_t. It is worth mentioning that one has to harvest the resource before using it as the input factor. [4]

For this reason, we assume that the agents harvest a specific stock N_t of a natural resource in every period of time t. The maximum carrying capacity of our resource is defined by M, which is obviously time independent and exogenously given, obviously. As usual in resource economics, we consider a logistic growth of our resource N_t as follows:

$$\frac{dN_t}{dt} = \xi N_t \left[1 - \frac{N_t}{M}\right] - \psi E(N_t). \tag{1}$$

Equation 1 is often called the "Schaefer equation", which is gathered from the Gordon-Schaefer model (Gordon, 1954), which is often used to discuss issues stemming from resource economics. As we can see from equation 1, we assume that a

[3] See for instance (Dasgupta and Heal, 1979).

[4] The cost of harvesting are not considered in this model.

fixed quantity of the natural resources is removed in every period of time, for example in every year. Furthermore, ξ represents the exogenous growth rate of the resource as mentioned before and $E(N_t)$ represents the aggregate harvesting function, which depends on the stock N_t. ψ represents the exogenous catchability coefficient, which is of no further interest.

To make the discussion easier, we restrict ourselves in the following on the two strategy case $i = \{h, j\}$ without loss of generality. In this manner, we create a channel to introduce agent specific heterogeneity (Noailly et al., 2003).

The two strategies can be formulated as follows: the first strategy h we label the "green strategy", which means that this strategy is less productive but less resource intensive than the strategy j, $h \neq j$. The other strategy j we call the "black strategy", because it is more resource intensive but more productive than the first one. Hence, one can conclude that

$$E(\cdot)_h C(\cdot)_h > E(\cdot)_j C(\cdot)_j \tag{2}$$

must hold. In equation 2, C_i stands for the cost of production and E_i stands for the effort of strategy i, which is given exogenously.[5]

We further assume that we can use the resource as input factor directly, which means that we do not include an intermediate good production sector in the model. The implication of this assumption is that the cost of production must include the cost of harvesting and furthermore, the cost of harvesting must equalize with the cost of production since we do not include other costs of production in our analysis. Subsequently, in the following we only use the terminology "cost of production".

As mentioned before, we assume a linear production function with input N_t. Thus, one can write for the production in period t, $\forall t$:

[5] Unless it is necessary, we leave the time index t for convenience.

$$F_t(N_t)^i = E_i N_t,\ i = \{h, j\}\ , \forall t \tag{3}$$

As usual in resource economics, we define the cost per product as $c_t^i(N_t) \equiv \frac{C_t^i}{N_t+1}$, which implies the more resource intensive the production, the more expensive is the extraction of a fraction of the stock N_t in the next period. We, therefore, assume an implicit forward looking agents behavior. Of course, if $N_t = 0$, then $C_t^i(0) = 0,\ \forall\, t$ per assumption. Consequently, N_t has to be treated as a necessary input factor for production. Again the reader has to bear in mind that the only purpose of the above mentioned assumption is to endogenize the production decision via endogenous cost of production.

With the above mentioned assumption it is now possible to deduce the profit per period t of the agents strategies $i = \{h, j\}$ which depends exclusively on the stock of the resource N_t, as one easily can derive from the next equation:

$$\Pi_t^i \equiv E_i N_t (p - c_i),\ i = \{h, j\}. \tag{4}$$

From that equation it is easy to obtain the profit per unit of strategy i in period t π_t^i as follows:

$$\pi_t^i \equiv \left[\frac{\Pi_i}{F_t^t}\right] = p - c_i, \tag{5}$$

where p stands for the exogenous price level.

Additionally, we assume similar to (Noailly et al., 2003) that the aggregate harvesting function is a convex combination of the single harvesting functions F_i. If we assume during a certain period of time a fraction β of the total population $\mathcal{B}\,, \beta \subseteq \mathcal{B}$ explicitly decide to use the strategy i with $\sum_i s_i = 1$ we can formulate the aggregate harvesting function as

$$E(N_t) = \sum_i s_i \beta F_i(N_t). \tag{6}$$

s_i stands for the market share of using strategy i.[6] With the last paragraph we have described the production sector totally.

To sum up, the main purpose of this section is to describe the dependence of the evolution of the scarcer resource N in the production sector and the influences of the evolution of N_t on the cost of production C_i under usage a certain strategy i from a pool of strategies n. In the next section, we proceed with some comments concerning the market evolution in the model.

2.2 Evolution of the market share

To expand the dynamic dimension in the model, we assume that the market share s_i under usage of strategy i will change over time. Therefore, we have to acknowledge the time aspect in the expression of s_i. To model the dynamic dimension of s_i, we recur to some facts of the field of population genetics, on which evolutionary economics is mainly based.

In the year 1908, (Hardy, 1908) published a striking article, which can be treated as a cornerstone for mathematical orientated population genetics. In his article he assume that

- some genetic frequencies, which he labelled (q, p) of two allele of a certain gene position have not to be unchanged by reproduction over the generations, belong to a certain population. The implication of this assumption is that the possibility of selection is excluded.

- the probabilities of belonging to a certain genotype (x, y, z) is exclusively defined by the initial co-generation in the way that: $x = p^2,\ y = 2pq,\ z = q^2$.

(Fisher, 1930) formulated a general equation of population genetics which bases on the ideas of (Hardy, 1908):[7]

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[6] One can set the $\mathcal{B} = 2$ so that one strategy i corresponds to one agent in our two strategy case $h = i, j$.

[7] In the following paragraph we use i and j for the generations of genotypes.

$$\dot{x}_i = x_j \left(\sum_i \omega_{ij} x_j - \sum_{r,s} \omega_{r,s} x_r x_s \right), \tag{7}$$

with $\sum_i \omega_{ij} x_j$ as the fitness of reproduction of all genotypes $A_i A_j$, ω as the advantage of survival and $\sum_{r,s} \omega_{r,s} x_r x_s$ as the average fitness for all other genotypes from $A_r, A_s, \in \mathcal{H}$, $\{r, s\} \neq j$. The expression $\sum_j \omega_{ij} x_j - \sum \omega_{r,s} x_r x_s$ can be interpreted as the advantage of survival, as a consequence.

Next, we adopt the ideas of the (Fisher, 1930) equation on our problem of how to model the market share evolution. From the above gained facts, we can conclude that s_i depends solely on the comparison between the fitness f_i and the average fitness $\bar{f}$ of the chosen strategy i of an agent.

In general, fitness depends on a n-dimensional vector $\mathbf{s}$ which contains the relative frequency of all possible replicators. Accordingly, one can write:

$$\dot{s}_i = s_i [f_i(\mathbf{s}) - \bar{f}(\mathbf{s})]. \tag{8}$$

Now we are able to adopt this general equation for our purpose. If we assume that the rate of capacity enlargement g_i corresponding to the usage of strategy i is positive related to the profit per unit π_t^i, we can write:

$$g_i = \gamma \pi_i = \gamma (p - c_i) = \gamma \left[p - \frac{C_i}{N+1} \right] \tag{9}$$

with the reaction coefficient $\gamma > 0$.

Next we define the average cost per product as $\bar{c} = \sum_n s_i c_i$, the average capacity enlargement rate or the average growth rate of the population of firms using a profitable strategy as $\bar{g} = \sum_n s_i g_i$ and set $f_i(\mathbf{s}) = g_i(\mathbf{s})$. Together with the derived equation we can write

$$\dot{s}_i = s_i (g_i - \bar{g}). \tag{10}$$

After doing some algebra, we can rewrite equation 10 together with equation 8 and equation 9 for strategy h for instance as follows:

$$\dot{s}_h = \gamma s_h(\bar{c} - c_h) = \gamma s_h \left(\frac{\bar{C} - C_h}{N+1} \right) = (1 - s_h) \left(\frac{C_j - C_h}{N+1} \right), \tag{11}$$

whereas we set in the last step without loss of generality $\gamma = 1$. What can we gain from this last derived equation? First, the market share s_h of using strategy h is directly linked to the cost of production C_i under usage strategy i for $i = \{h, j\}$. We can conclude that:

$$\dot{s}_h \left\{ \begin{array}{c} > 0 \\ < 0 \\ = 0 \end{array} \right\} \text{for} \left\{ \begin{array}{c} C_h < C_j \\ C_h > C_j \\ C_h = C_j \end{array} \right\}. \tag{12}$$

This holds also for c_h, as one can easily demonstrate.

Second, by a given stock of N_t the evolution of strategy s_i depends only on the cost relation to a competing strategy j. If the cost difference $\Delta(C) \equiv C_h - C_j = 0$, then, the agents should be indifferent between these two strategies from the pool n by a given level of E_i or the agents have no incentive to change their strategy. Otherwise we have a strictly dominating strategy $s_h \succ s_j$ for $\Delta(C) \equiv C_h - C_j < 0$ et vice versa.

Third, one can derive the following relationship of s_h and C_h for a given stock of N_t:

$$\frac{\partial \dot{s}_h}{\partial C_h} < 0 \tag{13}$$

and

$$\frac{\partial \dot{s}_h}{\partial C_j} > 0. \tag{14}$$

Subsequently, for a steady state of s_h, s_h^* (which means that $\dot{s}_h = 0$) we can conclude:

$$\dot{s}_h = 0 \Leftrightarrow s_h^* = 1 \wedge C_h^* = C_j^* \text{ for } s_h^* \in (0, 1). \tag{15}$$

Fourth, it is worth deducing a relationship between the evolution of the market share $\dot{s}_h$ and the stock of N_t:

$$\frac{\partial \dot{s}_h}{\partial N_t} \left\{ \begin{array}{ll} < 0 & \text{for } C_j > C_h \\ > 0 & \text{for } C_j < C_h \\ = 0 & \text{for } C_j = C_h \end{array} \right\} . \tag{16}$$

The main purpose of this section was to give an guess on how the evolution of the market share s_i depends on the resource N_t and the cost of production C_i. Until now, we have an imagine about the market structure and the production sector. In the next section, we want to introduce technological progress in the model. Thus, we set the main focus on the evolution of the cost structure $\dot{C}_i$.

2.3 Technological progress and market selection

As mentioned before, we model agent specific heterogeneity via different cost regimes at the beginning of the production period in $t = 0$. It is plausible to assume that agents invest in a less cost intensive producing technology.

In this way, we give a further dimension of what we call the structural dynamic aspect in the model. It is easy to see why. For a moment let us assume that an agent uses a strategy h with cost of production tending zero in the long run. Compared to a competing strategy j it is straightforward that this strategy j is ruled out of the strategy field of all agents which are producing in the market if $C_j \mapsto \bar{C} > 0$ for $t \mapsto \infty$. Hence, we have, looking at our previous results, a strictly dominating strategy h which monopolized the market. Consequently, the market itself is monopolized because the market share s_h by using strategy h tends to the value 1 in the long run. That‘s exactly the link between the existence of technology progress and how it influences the market structure in the long run. Of course, in the short run one can imagine some turbulences a propos the market evolution. This observation covers the industry life cycle assumptions (Malerba and Orsenigo, 1993).

Herewith, we want to create a direct link from the model to some ideas of Schumpeter on the subject of the dimension of structural dynamics and volatility. A wide known thesis proposed by Schumpeter is that creative destruction is a necessary condition for innovative firms. Of course, such firms have to dispose of financial potential to invest in R&D. Schumpeter assumes that the financial potential of firms is positive correlated to the market power of the firm. Thus, we realize a process of creative destruction mainly driven by R&D investments of innovative firms. The implication is that monopoly power is a necessary condition to create incentive for investments into in a technology which itself drives technological progress. (Neumann et al., 1982) conclude "that larger firms ... acquire smaller firms in order to exploit the innovative potential originated in these firms"[8].

It is also obvious that a trade off between the static and dynamic characteristics of competition exists. On the one hand one can realize extra rents from monopoly power which ensure growth, on the other hand, we have to acknowledge an allocative loss of efficiency. Consequently, the size of firms, the degree of concentration and innovativeness are positive correlated. From this follows that a higher degree of concentrated industries must exhibit higher growth rates (Schumpeter, 1942). The "Schumpeterian hypothesis II" is a major element in the frame work of the models of endogenous growth, which are mainly promoted by (Aghion and Howitt, 1992).

On contrary, (Arrow, 1962) showed in his article that the incentive of investing in R&D is negative correlated with the market power of an industry. He compares the gain of a cost reduction process innovation in a competitive world with the additional gain of cost reduction process innovation in a concentrated industry. He shows that the increase of profit in a competitive world is larger than in a monopoly.

The implication of the above mentioned is that many and small firms are more innovative in a competitive world, while few but large firms are more innovative in a concentrated world.[9]

It is short mentioned that a mass of literature exists which aims to test the Schumpeter hypothesis empirically.[10]

[8](Neumann et al., 1982), p. 135.

[9]Refer to (Acs and Audretsch, 1987).

[10]For the relationship between the size of firms, the degree of concentration and innovativeness

The question arise how we can integrate some facts of the Schumpeter hypothesis in our model? If one recalls our formulation of equation 11, we can derive a direct link between market share development and the degree of concentration. It is straightforward that the higher the market share s_i is the more successful the strategy i must be. On that account, we can suppose that the incentive to invest in a more successful strategy i is higher compared to an inferior strategy j. It follows that the size of a firm and the power of investment are striking factors for the market structure and as well for the evolution of the market share s_i. Therefore, they both are positive correlated to the success of a strategy i.

The implication is, if we follow (Phillips, 1971), that the positive correlation of the firm size and the innovativeness follows a "success-breeds-success" hypothesis.[11] This implies that a positive dependence of successful innovation activity in the current period and investment endeavours in the next periods exists. Following (Cantner and Hanusch, 1998) or (Malerba and Orsenigo, 1993) this interpretation is in line with the so called "Schumpeterian II" hypothesis. On the other site, one could argue that smaller firms are more innovative because their behaviour regarding to investment decisions is more flexible (Malerba et al., 1997). This view equivalent to the "Schumpeterian I" hypothesis.[12]

It is worth noting that the "Schumpeterian I" and the "Schumpeterian II" hypothesis are common patterns which can occur both through the industry lifecycle, whereas the early stage is characterized by the "Schumpeterian I" hypothesis, while the later periods are more in line with "Schumpeterian II". The implication is that a Schumpeter we regime should be more volatile than the "Schumpeterian II" hypothesis but stable. It is common measuring the stability with a so called "instability index"

refer to (Cohen and Levin, 1989) and (Kamien and Schwartz, 1975) and for the relationship between the firm size and innovativeness refer to (Frisch, 1993). For German data refer to (Neumann et al., 1982), (Entorf, 1988), (Kraft, 1989), and to (Bertschek and Entorf, 1996) for Belgian, German and French data.

[11] See for instance (Flaig and Stadler, 1994) who have found an empirical confirmation of the success-breeds-success hypothesis for West-Germany using a German panel data set.

[12] For instance refer to (Acs and Audretsch, 1987), (Malerba and Orsenigo, 1993) and (Malerba et al., 1997).

(Hymer and Pashigan, 1962) which is defined as:

$$\Im = \sum_{i}^{n} |s_t^i - s_{t-1}^i|, \tag{17}$$

with s_t^i as the market share of strategy i at time t.

To catch this interesting ideas, we follow (Malerba and Orsenigo, 1993), (Malerba et al., 1997) and (Cantner and Hanusch, 1998) and create a direct link from the "Schumpeterian I" hypothesis to increasing returns of investing in a new, less cost intensive, technology. Instead of the "Schumpeterian I" hypothesis, the "Schumpeterian II" hypothesis is directly associated with the assumption of decreasing returns of investing in a new cost reduction technology. Additionally, we incorporate the case of constant returns to scale for the sake of completeness. The latter case is extensively discussed by (Metcalfe, 1994). Before we proceed, we have to point out that we model technological progress in the sense of total cost reduction over time, not in the sense of (Metcalfe, 1994). In contrast to (Metcalfe, 1994), the cost of production C_i are endogenous in this case.

In this way, it is quite easy to give an idea of how the evolution of the cost of production C_i for period t should be modeled:

$$\dot{C}_i = \begin{cases} -\theta C_i \\ -\theta C_i s_i \\ -\theta C_i (1 - s_i) \end{cases} . \tag{18}$$

Please note that $\theta \in (0,1)$ represents the rate of technological progress in cost reduction. We make the assumption that the more one invests in the technology the faster the cost reduction progress associated with an large value of θ proceeds. The progress is fastest for θ tending to 1. As one can see from equation 18, the first line represents the case of constant returns to scale, the second line the case of increasing returns to scale and the last line the case of decreasing returns to scale. In the next chapter, we give the formulation of a model of market selection which is driven by structural dynamics.

2.4 Setup of the model

After describing the ideas of integrating technological progress in the model, we am now able to formulate a comprehensive model which integrates the following three aspects:

1. Resource dynamics N_t, which influence

2. the selection of a producing strategy s_i via an endogenous cost structure C_i,

3. which is itself driven by technological progress θ in the cost structure.

These three points can be summarized in a more mathematical manner as follows:

$$\left\{ \begin{array}{lcl} \dot{N} & = & \xi N \left[1 - \frac{N}{M}\right] - \psi\beta \left[\sum_n s_i F_i(N)\right] \\ \dot{s}_i & = & s_i(c_i - \bar{c}) \\ \dot{C}_i & = & \left\{ \begin{array}{c} -\theta C_i \\ -\theta C_i s_i \\ -\theta C_i(1 - s_i) \end{array} \right. \end{array} \right. . \tag{19}$$

Again, the first line of the system of equations 19 represents the evolution of the resource N_t, the second equation gives an impression of how the market share s_i is influenced by using strategy i. Whereas the last line distinguish between the different kinds of returns to scale respective to the investment into a new cost reduction technology.

After we have a created the setup of our model, the next step is to solve it and to examine its behaviour especially in the short run. The emerging question is how to study the dynamic of such a system. As can be seen from above, we are confronted with an systems containing non-linear differential equations. Therefore, in the next section we have to add some comments in respect of the dynamic behaviour of such a system.

2.5 Dynamic behaviour of the model

To discuss the dynamic behavior of our system, we have first to investigate, whether a steady state in the sense of a long run equilibrium exists. This is equivalent to the postulation that the partial derivatives over time of $\dot{N}_t$, $\dot{s}_t$ and $\dot{C}_t$ must be zero. Hence, we can formulate the following proposition:

Proposition 1: On behalf of the assumption that the partial derivatives of $\dot{N}_t$, $\dot{s}_t$ and $\dot{C}_t$ exist and that $\dot{N}_t = 0$, $\dot{s}_t = 0$ and $\dot{C}_t = 0$ hold simultaneously $\forall t$, the system 19 has a unique steady state vector $\mathbf{S}$, which contains N^*, s_i^* and C_i^* in the long run.□

Next, we give a brief sketch of how to proof of proposition 1.

Proof 1: An optimal steady state vector exists, if and only if $\dot{N} = \dot{s}_i = \dot{C}_i = 0$ holds. This is realized, if

$$\begin{cases} \xi N^* \left[1 - \frac{N^*}{M}\right] & = & \psi\beta \left[\sum_n s_i^* F_i(N^*)\right] \\ 0 & = & s_i^*(c_i^* - \bar{c}) \\ 0 & = & \begin{cases} -\theta C_i^* \\ -\theta C_i^* s_i^* \\ -\theta C_i^*(1 - s_i^*) \end{cases} \end{cases} . \tag{20}$$

As one can see from the system of equations 19, the equation for the evolution of C_i is influenced only by s_i but not by N for the increasing and decreasing returns to scale case each. For the constant returns to scale case the evolution of C_i is purely autonomous in the sense that it is not influenced by N_t or s_i. As a consequence the value of θ is an important determinant for the market structure evolution s_i. Thus, we can find the following:

1. Assume now, that a value of θ exists which is greater than a threshold value of θ, $\tilde{\theta}$ and near to a maximum value of θ, called θ_{max} so that $\theta_{max} > \theta \gg \tilde{\theta}$ holds. Then, speed of cost reduction is very fast and as a result after a short period of time $\dot{s}_i = 0$. Additionally, from the second and third line of the steady state system follows immediately that $C_h^* = C_j^* = 0$ and $s_i^* \in (0, 1)$, for every case of returns to scale assumption. We obtain $N^* = M\left[1 - \frac{\psi\beta[\sum_n E_i s_i^*]}{\gamma}\right]$.

2. Further, assume that a smaller value of θ exists which is near to the minimum value of θ, called θ_{min}, and smaller than the threshold value $\tilde{\theta}$. Then $\tilde{\theta} \gg \theta > \theta_{min}$ holds, obviously. Accordingly, technological progress is very slow. For $C_h(0) > C_j(0)$[13] in $t = 0$ follows

   (a) for the constant returns of scale case: $C_h^* = C_j^* = 0$ which means that $s_h^* = 0$. In addition, we obtain $N^* = M\left[1 - \frac{\psi\beta E_j}{\gamma}\right]$.

   (b) for the increasing returns to scale case: $C_h^* \in \mathbb{R}_+ \wedge C_j^* = 0$ which means that $s_h^* = 0$. Once again, we obtain $N^* = M\left[1 - \frac{\psi\beta E_j}{\gamma}\right]$.

   (c) for the decreasing returns to scale case with the further assumption that a $\epsilon \mapsto 0$ exists, for which one assume that $\epsilon < \tilde{\epsilon} \equiv |\theta - \theta_{min}|$: $C_j^* \in \mathbb{R}_+ \wedge C_h^* = 0$ which means that $s_h^* = 0$. Once more, we obtain $N^* = M\left[1 - \frac{\psi\beta E_j}{\gamma}\right]$. From point 2 of proof 1 follows that $\tilde{\theta} \gg \epsilon \gg \tilde{\epsilon}$ must hold. For that reason, we obtain $N^* = M\left[1 - \frac{\psi\beta[\sum_n E_i s_i^*]}{\gamma}\right]$ for $s_i^* \in (0, 1)$.

3. If 1 or 2 of proof 1 holds, then $\exists\, N^* \in \mathbb{R}_+ \setminus \{0\}$.

4. If one assumes $N^* = 0$, then a set $\mathcal{I}$ of degenerated equilibria is realized for $s_i^* \in (0, 1)$ and $C_h^* = 0$ because ane degree of freedom is left to set s_i^*.

■

Hence, we have several steady states values or a set of steady state values to take into consideration, which could all exist. But what follows from proof 1 intuitively? First, the dynamic is only driven by the parameter θ, which is purely exogenous per assumption. Consequently, we obtain different scenarios regarding to our market structure depending only on the parameter of technological progress which is not explained by our model.[14]

Second, if one sets $\theta = 0$ we obtain a two dimensional system consisting only in the development of N_t and s_i.

[13]Of course, one can assume $C_h(0) < C_j(0)$. If $C_h(0) = C_j(0)$ we cannot observe any dynamic of s_i and N_t right from $t = 0$. Then $s_i(0) = s_i^*$ follows.

[14]Technological progress falls like "manna from heaven". See for instance (Frenkel and Hemmert, 1999), p. 113.

Thus, this model can be described as a variant of the model of (Noailly et al., 2003). On the other hand, if we handle N_t as constant $N_t = N$, $\forall t$, we obtain a market structure model similar to (Cantner and Hanusch, 1998).

Now we will proceed with the stability analysis of the model.

It does not make sense to discuss further topics on a underlying a model which is not stable, of course.

2.6 Stability analysis

As mentioned before, our system 19 consists in non linear differential equations. To deduce the local stability of our economic variables of interest N_t, s_i and C_i, we should linearize the system 19 around the steady state values N^*, s_i^* and C_i^*.

To make some comments concerning to the dynamical behavior of our system 19, we have to derive the Jacobi-Matrix and its corresponding Eigenvalues.

As a first step, it is common to write for an n-dimensional non linear system in general:

$$\dot{\mathbf{x}} = F(\mathbf{x}_t, \mathbf{v}_t). \tag{21}$$

Hereby $\mathbf{F}(\cdot)$ is a $(n \times 1)$-dimensional vector containing n vectors $f_n(\cdot)$ of non linear functions, $\dot{\mathbf{x}}$ is a $(n \times 1)$- dimensional vector which contains the partial derivatives of x with respect to t and $\mathbf{v}_t$ is a $(n \times 1)$-vector of time dependent values. For our purpose we set $\mathbf{v}_t = \mathbf{0}$ without loss of generality. Therefore, we can rewrite equation 21 and obtain[15]:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2, ..., x_n) \\ f_2(x_1, x_2, ..., x_n) \\ \vdots \\ f_n(x_1, x_2, ..., x_n) \end{bmatrix}. \tag{22}$$

To discuss the dynamic behaviour of our system in a ϵ-neighbourhood of the steady state values $x^* = [x_1^*, x_2^*, ..., x_n^*]'$, we have to linearize our system around its steady state vector x^* using a Taylor expansion or approximation, respectively.

[15] We set the subscripts t in notational form to indicate time dependent variables.

The intuition behind a first order Taylor expansion is to express the deviations Δ of the variables of interest $\mathbf{x}$ from their steady state values $\mathbf{x}^*$. Thus, we get:

$$\begin{aligned}
\dot{x}_1 &= f^1(x^*) + f^1_{x_1}(x^*)(x_1 - x_1^*) + ... + f^1_{x_n}(x^*)(x_n - x_1^*) + \vartheta_1, \\
\vdots \quad &\vdots \quad \vdots \\
\dot{x}_n &= f^n(x^*) + f^n_{x_1}(x^*)(x_n - x_1^*) + ... + f^n_{x_n}(x^*)(x_n - x_n^*) + \vartheta_n.
\end{aligned} \tag{23}$$

If $|x - x^*| \to \epsilon$ then $\vartheta_i = 0,\ i = \{1, 2, ..., n\}$. The advantage of the Taylor approximation in the neighbourhood of the steady state is that the first elements of every equation i vanish because of the existence of a steady state. The implication is that $\dot{x}_i = 0\ , \forall\, i$.

In matrix algebra we can write:

$$\frac{\partial \Delta \mathbf{x}}{\partial t} = F(\mathbf{x}^* + \Delta \mathbf{x}). \tag{24}$$

Or, if we apply the Taylor expansion on $F(\cdot)$ around the steady state values $\mathbf{x}^*$, one can derive:

$$\frac{\partial \Delta \mathbf{x}}{\partial t} = F(\mathbf{x}^*) + \frac{\partial F(\cdot)}{\partial \mathbf{x}}|_{\mathbf{x}=\mathbf{x}^*}[\mathbf{x} - \mathbf{x}^*] + \vartheta(\Delta \mathbf{x}), \tag{25}$$

whereas the residual vector $\vartheta(\Delta \mathbf{x})$ can be treated as a redundant variable, as mentioned before. It is easy to see that we have to compute n-partial derivatives for each $f_i(\cdot)$ such we get at all together $n \times n$ derivatives for the matrix $\mathbf{F}(f_1, f_2, ..., f_n)$. Subsequently, we may concentrate our facts and write in matrix algebra in a more convenient manner:

$$\dot{\mathbf{x}} = A\overline{\mathbf{x}} \tag{26}$$

with $\overline{\mathbf{x}} \equiv \mathbf{x} - \mathbf{x}^*$. Here we define

$$A \equiv \left[\frac{\partial F(\cdot)}{\partial \mathbf{x}}|_{\mathbf{x}=\mathbf{x}^*}\right] = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} & \cdots & \frac{\partial \dot{x}_1}{\partial x_n} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} & \cdots & \frac{\partial \dot{x}_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \dot{x}_n}{\partial x_1} & \frac{\partial \dot{x}_n}{\partial x_2} & \cdots & \frac{\partial \dot{x}_n}{\partial x_n} \end{bmatrix}. \tag{27}$$

One can verify that in expression 27 A is called the Jacobian of the system. Herewith, we have transformed a non linear system into a linearized non linear system, which is not homogeneous. It is known that for linear systems stability can be proofed applying the Jacobian and their corresponding Eigenvalues. This strategy is also valid for linearized homogeneous systems. But before we proceed, we have to make a proposition with respect to the relationship between homogeneous and inhomogeneous systems to proof stability.

Proposition 2: All solutions, which can be derived from a linearized inhomogeneous system, are stable or attractive or even asymptotically attractive, if and only if the trivial solution of the corresponding homogeneous system is stable or attractive or even asymptotically attractive.□

The proof is rather short and requires nothing else then some facts of variation computation. We note down the proof for an n-dimensional case[16]:

Proof 2: All solutions $\mu : (T, \infty) \rightarrow \mathbb{R}^n$ of an system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}+\mathbf{b}$ consist in the same differential equation of defective movement $\dot{\mathbf{y}} = [\mathbf{A}(\mathbf{y}+\mu)+\mathbf{b}]-[\mathbf{A}\mu+\mathbf{b}] = \mathbf{A}\mathbf{y}$, which is the same analytical expression as for the corresponding homogeneous differential equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. ■

On the basis of the proposition 2 we know that is always possible to check the stability of a inhomogeneous system by its homogeneous counterpart. Thus, we can refer to the matrix $\mathbf{A}$ in equation 27.

Furthermore, we have to make a proposition on the subject of the Eigenvalues conditions regarding their stability. Consequently, we have to formulate an additional proposition 3 for this purpose:

Proposition 3: All solutions of a homogeneous system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ are

a) stable, if all Eigenvalues, which can be derived from a matrix $\mathbf{A}$, are real ≤ 0 and those Eigenvalues who are zero must be semisimple.

b) asymptotically stable, if all Eigenvalues have an negative real part. If ρ_{max} is the maximum of the real part of the Eigenvalues, then, a constant $C =$

[16]One can refer to (Aulbach, 2004) for the 1×1-scalar case.

$C(\alpha) \geq 1$ with $||exp(At)|| \leq K\, exp(-\alpha t)$,$\forall t \geq 0$ exists for every positive α with $\rho_{max} < -\alpha < 0$.

□

Unlike of proof 1 or proof 2, it is not easy to give a similar simple proof for proposition 3. We, therefore, refer to (Aulbach, 2004), p. 331. It is straightforward that proposition 3 has to be considered as a striking element of our analysis.

Now we are prepared mathematically and can apply the above derived system to our equations of interest in 19. In the following, we set $\mathbf{x} = [x_1, x_2, x_3] \equiv [N, s_1, C_1]$, $i = \{h, j\} = \{1, 2\}$ and $A \equiv Jac$. One should notice and, for that reason, we have to point out that this system is necessary to describe our model for the two strategy case $i = \{1, 2\}$. We must not include s_2 because $\sum_i s_i = 1$ on aggregate and in consequence one can derive C_2 easily. As a result, we have reduced our problem from the space $dim\, [5 \times 5]$ to the space $dim\, [3 \times 3]$.

After the linearisation of our system of equations 19 the following Jacobi-Matrix results:

$$Jac^{crs} \equiv \begin{bmatrix} \frac{\partial \dot{N}}{\partial N} & \frac{\partial \dot{N}}{\partial s_1} & \frac{\partial \dot{N}}{\partial C_1} \\ \frac{\partial \dot{s_1}}{\partial N} & \frac{\partial \dot{s_1}}{\partial s_1} & \frac{\partial \dot{s_1}}{\partial C_1} \\ \frac{\partial \dot{C_1}}{\partial N} & \frac{\partial \dot{C_1}}{\partial s_1} & \frac{\partial \dot{C_1}}{\partial C_1} \end{bmatrix} =$$

$$= \begin{bmatrix} -\gamma(\frac{N^*}{M}) & -\psi\beta N^*(E_1 - E_2) & 0 \\ -s_1^* \frac{(C_1^* - C_2)^*(s_1^* - 1)}{(N^*+1)^2} & \frac{(C_1^* - C_2^*)(2s_1^* - 1)}{N^*+1} & \left[\frac{s_1^* - 1}{N^*+1}\right] s_1^* \\ 0 & 0 & -\theta \end{bmatrix}. \quad (28)$$

Next, we have to evaluate the Jacobian at their steady state values for N^*, s_1^*, C_1^* and C_2^* for $\theta \in (0, 1)$. It is easy to verify that we have to check the stability for every case of returns to scale. That work is done in the following.

First, we examine the constant returns to scale case. Refering to point 1, 2a) and 4) of proof 1 we can conclude that we get three different versions of the Jacobian Jac_f^{u}[17] matrix:

[17]The subscript f denotes to the numeration of proof 1, whereas the superscript u denotes to the cases of returns to scale: crs stands for the constant returns to scale case, irs stands for the increasing returns to scale case and drs stands for the decreasing returns to scale case.

$$Jac_1^{crs} = \begin{bmatrix} -\gamma(\frac{N^*}{M}) & -\psi\beta N^*(E_1 - E_2) & 0 \\ 0 & 0 & \left[\frac{s_1^*-1}{N^*+1}\right] s_1^* \\ 0 & 0 & -\theta \end{bmatrix} \tag{29}$$

$$Jac_{2a)}^{crs} = \begin{bmatrix} -\gamma(\frac{N^*}{M}) & -\psi\beta N^*(E_1 - E_2) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\theta \end{bmatrix} \tag{30}$$

$$Jac_4^{crs} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & (s_1^* - 1)s_1^* \\ 0 & 0 & -\theta \end{bmatrix}. \tag{31}$$

To see whether the equilibria points are stable or not we have to deduce the characteristic vector of the Eigenvalues for each equilibrium point. For convenience, we stack the Eigenvalues in a vector Ψ_f^u[18] each. Hence, one get

$$\Psi_{1,2a)}^{crs} \equiv \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\gamma\frac{N}{M} \\ -\theta \end{bmatrix} \text{ and}$$

$$\Psi_4^{crs} \equiv \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\theta \end{bmatrix}.$$

The first thing to notice is that we only obtain real Eigenvalues for the constant returns to scale case. Consequently, complex Eigenvalues are ruled out because of proposition 4 which is presented right below.

[18]The subscript f denotes to the numeration of proof 1, whereas the superscript u denotes to the cases of returns to scale: crs stands for the constant returns to scale case, irs stands for the increasing returns to scale case and drs stands for the decreasing returns to scale case.

Proposition 4: Assume that $\lambda^3 + [-tr(Jac)]\lambda^2 + \Omega\lambda + (-|Jac|) = 0$ with $\Omega \equiv \begin{vmatrix} \frac{\partial \dot{x}_{22}}{\partial x_{22}} & \frac{\partial \dot{x}_{23}}{\partial x_{23}} \\ \frac{\partial \dot{x}_{32}}{\partial x_{23}} & \frac{\partial \dot{x}_{33}}{\partial x_{33}} \end{vmatrix} + \begin{vmatrix} \frac{\partial \dot{x}_{11}}{\partial x_{11}} & \frac{\partial \dot{x}_{13}}{\partial x_{13}} \\ \frac{\partial \dot{x}_{31}}{\partial x_{31}} & \frac{\partial \dot{x}_{33}}{\partial x_{33}} \end{vmatrix} + \begin{vmatrix} \frac{\partial \dot{x}_{11}}{\partial x_{11}} & \frac{\partial \dot{x}_{12}}{\partial x_{21}} \\ \frac{\partial \dot{x}_{21}}{\partial x_{21}} & \frac{\partial \dot{x}_{22}}{\partial x_{22}} \end{vmatrix}$ of the $dim\,[3 \times 3]$ case holds. Then, a necessary and sufficient condition for two imaginary Eigenvalues with real part zero ($\pm\omega i$, $i = \sqrt{-1}\omega$, $\omega \neq 0$) of the characteristic equation is:

1. $\Omega > 0$ and
2. $\Omega[-tr(Jac)] + |Jac| = 0$.

□

A proof of this proposition can be found in (Asada and Semmler, 1995).

Second, the Eigenvalues for the first two equilibria coincide because the steady state values for C_1^* and C_2^* must reach the value 0 in each case, if we assume a cost reduction driven by a technological progress. Additionally, as can be seen from the Jacobian a multiplicative relationship between the cost difference $\Delta C \equiv C_1^*$-C_2^* and s_1^* exists.

Refering to proposition 3 b), we have to know whether the Eigenvalues are semisimple. Here we give the following two definitions:

Definition 1: If λ is called a i-time root of a characteristic polynomial $i \in n$, then $b_a(\lambda) \equiv i$ is called the algebraic multiplicity of the Eigenvalue. The dimension $b_{geo}(\lambda)$ of the corresponding Eigenspace, which means the maximum number of orthogonal Eigenvectors, is called the geometric multiplicity of the Eigenvalue. Therefore, the relation $1 \leq b_{geo}(\lambda) \leq b_a(\lambda) \leq n$ must hold. If $b_a(\lambda) = b_{geo}(\lambda)$, then, the Eigenvalue is called semisimple.

If one is familiar with linear algebra, it is quiet obvious that we need a definition of the Eigenspace before we continue because its dimension is the geometric multiplicity of the Eigenvalue.

Definition 2: If $A \in \mathbb{C}^{n \times n}$ and if λ is a Eigenvalue of A, then the kernel $\mathcal{N}(A - \lambda I)$ is called the Eigenspace of λ and its dimension is called the algebraic multiplicity $\wp$ of λ. It follows that: $\wp \equiv b_a(\lambda) = dim\,[\mathcal{N}(A - \lambda I)] = n - Rank(A - I)$.

Definition 2 is only valid for the complex space ex ante. Because of that, we have to make some remarks on the case of real Eigenvalues, if we want to apply proposition 3 b) to our problem.

Two annotations:

- If $\lambda \in \mathbb{R}$, and because of $A \in \mathbb{R}^{n \times n}$ we obtain real Eigenvalues from $(A - \lambda I)x = 0$ and the base of an Eigenspace regarding to λ can be obtained from $\wp$ real Eigenvectors.

- If $Im(\lambda) \neq 0$, then, all Eigenvectors are real complex. With λ, another vector $\bar{\lambda}$ is a valid Eigenvalue because it follows from $\Psi_A(\lambda) = 0$ that $\Psi_A(\lambda) = \Psi_A(\bar{\lambda}) = 0$ because Ψ_A is treated as a real polynomial.

Next, we can use some useful facts from linear algebra concerning the relationship of the Jacobian and the Jordan block form notation. This will help us to proof the Semisimplicity of the Eigenvalues.

Proposition 5: Assume that a given vector of real Eigenvalues exists. Further assume that the Jacobi-Matrix can be expressed by the Jordan normal form: $J = diag(J_1, ..., J_p)$, whereas J_i[19] represents the matrix containing all p Jordan blocks for the i-th Eigenvalue λ_i for $i = \{1, ..., p\}$. Then, a real Eigenvalue is called semisimple, if and only if the so called Jordan block J can be represented as a diagonal matrix or more precisely, for a real Eigenvalue λ_i one can write $J_i = diag(\lambda_i, ..., \lambda_i)$. Thus, $D = T^{-1} Jac\, T$ must equal J.$\square$

A proof of this proposition can be found in (Aulbach, 2004), p. 241. Note, that for a pair of real Eigenvalues λ_i we obtain $J_i \in \mathbb{R}^{b_a(\lambda) \times b_a(\lambda)}$.

As we can see from the system 29 to system 31, we cannot directly apply proposition 5, because Jac is not compatible with the Jordan normal form, obviously.

But we can easily transform the matrix A into a Jordan normal form. That implies that every symmetric Matrix A is diagonizable, if a regular matrix T exists, so that one can find a diagonal matrix $D = T^{-1}AT$. The matrix T consists of linear independent Eigenvectors which can be derived from the solution of the system

[19] Here we use the subscript i for the i-th real Eigenvalue.

$(A - \lambda I) = 0.$

We just have to make the transformation $D = T^{-1} Jac\, T$ and apply proposition 5 to D.

It is good to know the following definition:

Definition 2: If all Eigenvalues λ_i of a given matrix A are semisimple, then, the matrix A is called semisimple or diagonizable, which means that a matrix D exists. In the following, we show that D exists using definition 2. If we apply proposition 5 again to our 29 to system 31, we can see that the proposition holds. We exemplify this for the constant returns to scale case: The resulting Jordan blocks are obviously scalar for the equilibrium $f = 1$ and $f = 2a$). For equilibrium $f = 4$ we obtain for instance $J^u_{i,f} = J^{crs}_{1,4} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$[20] and $J_{2,4} = -\theta$.

As a result, $b_{geo}(\lambda_1) = b_a(\lambda_1) = b_{geo}(\lambda_2) = b_a(\lambda_2) = 2$ because $\wp = dim\mathcal{N}(A - \lambda_1 I) = n - Rank(A - \lambda_1 I) = dim\mathcal{N}(A - \lambda_2 I) = n - Rank(A - \lambda_2 I) = 2$ and $b_{geo}(\lambda_3) = b_a(\lambda_3) = 1$ because $\wp = dim\mathcal{N}(A - \lambda_3 I) = n - Rank(A - \lambda_3 I) = 1$. Thus the Eigenvalues are semisimple. From equilibrium $f = 4$ we know that 0 is a double root and $-\theta$ is a single root. Consequently, it follows that $D_f = D_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\theta \end{pmatrix}$[21] exists.

One can conclude that proposition 3 b) and 2 hold each. The result is that our system is stable, but not asymptotically stable for the constant returns to scale case. But it is not barred that the system converges before infinity, given it converges de facto. For this purpose it is sufficient to know that the system converges by a certain period of time $t \in \{0, 1, ..., T\}$.

Next, we have to do the same computations for the decreasing and increasing returns to scale case. Again, for this cases it can be shown that the system is stable, but

[20] The subscript f denotes to the numeration of proof 1, i denotes to the i-th real Eigenvalue, whereas the superscript u denotes to the cases of returns to scale: crs stands for the constant returns to scale case, irs stands for the increasing returns to scale case and drs stands for the decreasing returns to scale case.

[21] The subscript f denotes to the numeration of proof 1.

not asymptotically stable[22] by applying the above mentioned toolbox.

The Jacobian for increasing returns to scale is

$$Jac^{irs} \equiv \begin{bmatrix} \frac{\partial \dot{N}}{\partial N} & \frac{\partial \dot{N}}{\partial s_1} & \frac{\partial \dot{N}}{\partial C_1} \\ \frac{\partial \dot{s_1}}{\partial N} & \frac{\partial \dot{s_1}}{\partial s_1} & \frac{\partial \dot{s_1}}{\partial C_1} \\ \frac{\partial \dot{C_1}}{\partial N} & \frac{\partial \dot{C_1}}{\partial s_1} & \frac{\partial \dot{C_1}}{\partial C_1} \end{bmatrix} =$$

$$= \begin{bmatrix} -\gamma(\frac{N^*}{M}) & -\psi\beta N^*(E_1 - E_2) & 0 \\ -s_1^* \frac{(C_1^*-C_2)^*(s_1^*-1)}{(N^*+1)^2} & \frac{(C_1^*-C_2^*)(2s_1^*-1)}{N^*+1} & \left[\frac{s_1^*-1}{N^*+1}\right] s_1^* \\ 0 & -\theta C_1^* & -\theta s_1^* \end{bmatrix}. \tag{32}$$

If we evaluate the Jacobian at her steady state values for N^*, s_1^* and C_1^* for $\theta \in (0, 1)$, again, we obtain three different versions of the Jacobian matrix Jac_i, which each of them represent another equilibrium i:

$$Jac_1^{irs} = \begin{bmatrix} -\gamma(\frac{N^*}{M}) & -\psi\beta N^*(E_1 - E_2) & 0 \\ 0 & 0 & \left[\frac{s_1^*-1}{N^*+1}\right] s_1^* \\ 0 & 0 & -\theta s_1^* \end{bmatrix} \tag{33}$$

$$Jac_{2b)}^{irs} = \begin{bmatrix} -\gamma(\frac{N^*}{M}) & -\psi\beta N^*(E_1 - E_2) & 0 \\ 0 & \frac{-C_1^*}{1+N^*} & 0 \\ 0 & -\theta C_1 & 0 \end{bmatrix} \tag{34}$$

$$Jac_4^{irs} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & s_1^*(1 - s_1^*) \\ 0 & 0 & -\theta s_1^* \end{bmatrix}. \tag{35}$$

Once more, we define a column vector Ψ_f^u which contains the Eigenvalues for the first equilibrium as follows:

$$\Psi_1^{irs} \equiv \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\gamma\frac{N^*}{M} \\ -s_1^*\theta \end{bmatrix}.$$

[22] One can easily see that equilibrium 2b) for the decreasing returns to scale case is saddle path stable. The stability depends on the chosen starting values.

In the same way we stack the Eigenvalues for the second equilibrium in a column vector: $\Psi_{2b)}^{irs} \equiv \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\gamma\frac{N^*}{M} \\ -\frac{C_1^*}{1+N^*} \end{bmatrix}$.

And for the degenerated equilibrium we get: $\Psi_4^{irs} \equiv \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -s_1^*\theta \end{bmatrix}$.

If we now apply the pertinent propositions, which have been explained before, then, one can show that the equilibria obtained are stable for the increasing returns to scale case.

The Jacobian for decreasing returns to scale is

$$Jac^{drs} \equiv \begin{bmatrix} \frac{\partial \dot{N}}{\partial N} & \frac{\partial \dot{N}}{\partial s_1} & \frac{\partial \dot{N}}{\partial C_1} \\ \frac{\partial \dot{s_1}}{\partial N} & \frac{\partial \dot{s_1}}{\partial s_1} & \frac{\partial \dot{s_1}}{\partial C_1} \\ \frac{\partial \dot{C_1}}{\partial N} & \frac{\partial \dot{C_1}}{\partial s_1} & \frac{\partial \dot{C_1}}{\partial C_1} \end{bmatrix} =$$

$$= \begin{bmatrix} -\gamma(\frac{N^*}{M}) & -\psi\beta N^*(E_1 - E_2) & 0 \\ -s_1^* \frac{(C_1^* - C_2)^*(s_1^* - 1)}{(N^*+1)^2} & \frac{(C_1^* - C_2^*)(2s_1^* - 1)}{N^*+1} & \left[\frac{s_1^* - 1}{N^*+1}\right] s_1^* \\ 0 & \theta C_1^* & -\theta(1 - s_1^*) \end{bmatrix}. \quad (36)$$

The evaluation of the Jacobian at her steady state values for N^*, s_1^* and C_1^* for $\theta \in (0,1)$ results in:

$$Jac_1^{drs} = \begin{bmatrix} -\gamma(\frac{N^*}{M}) & -\psi\beta N^*(E_1 - E_2) & 0 \\ 0 & 0 & \left[\frac{s_1^* - 1}{N^*+1}\right] s_1^* \\ 0 & 0 & -\theta(1 - s_1^*) \end{bmatrix} \quad (37)$$

$$Jac_{2c)}^{drs} = \begin{bmatrix} -\gamma(\frac{N^*}{M}) & -\psi\beta N^*(E_1 - E_2) & 0 \\ 0 & \frac{C_2^*}{1+N^*} & 0 \\ 0 & 0 & -\theta \end{bmatrix} \quad (38)$$

$$Jac_4^{drs} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -(1-s_1^*)s_1^* \\ 0 & 0 & -\theta(1-s_1^*) \end{bmatrix}. \quad (39)$$

Finally, we obtain three vector Ψ_f^u containing the Eigenvalues of the three equilibria:

$$\Psi_1^{drs} \equiv \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\gamma(\frac{N^*}{M}) \\ -(1-s_1^*)\theta \end{bmatrix}$$

$$\Psi_{2c}^{drs} \equiv \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} -\gamma(\frac{N^*}{M}) \\ \frac{C_2^*}{1+N^*} \\ -\theta \end{bmatrix}$$

$$\Psi_4^{drs} \equiv \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ (s_1^*-1)\theta \end{bmatrix}.$$

As we can see from our results for the decreasing returns to scale case equilibria 1 and 4 are both stable. Equilibrium $2c$) is called saddle path stable, since two Eigenvalues are negative and one Eigenvalue is positive[23]. The results regarding to the algebraic and geometric multiplicity of the Eigenvalues are also drawn in table 1, which summarizes the stability discussion for all three cases.

[23]The stability of a saddle path stable equilibrium depends on the chosen starting values.

Case	λ_1		λ_2		λ_3	
	$b_a(\lambda_1)$	$b_{geo}(\lambda_1)$	$b_a(\lambda_2)$	$b_{geo}(\lambda_2)$	$b_a(\lambda_3)$	$b_{geo}(\lambda_3)$
Constant returns to scale (crs)						
Equilibrium 1	1	1	1	1	1	1
Equilibrium 2a)	1	1	1	1	1	1
Equilibrium 4	2	2	2	2	1	1
Increasing returns to scale (irs)						
Equilibrium 1	1	1	1	1	1	1
Equilibrium 2b)	1	1	1	1	1	1
Equilibrium 4	2	2	2	2	1	1
Decreasing returns to scale (drs)						
Equilibrium 1	1	1	1	1	1	1
Equilibrium 4	2	2	2	2	1	1

Table 1: Geometric and algebraic multiplicity of the obtained real Eigenvalues

3 Conclusion

The motivation for this contribution is based on the observation, that is not common to discuss the stability of equilibria in replicator dynamic modelling, especially, if this technique is employed in an evolutionary economic context.

As shown in this paper, often more than zero real Eigenvalues with value zero occur. From a technical point of view, this is a very interesting event. Unfortunately, the local stability criterion theorem defined by (Strogatz, 1994) and (Hilborn, 1994) cannot be applied to prove stability. To solve this problem, one can rely on the Semisimplicity of the Eigenvalues to proof, if the resulting equilibria are at least stable or not. Besides the (Strogatz, 1994) and (Hilborn, 1994) local stability theorem, this method provides an alternative procedure for stability analysis for rdm. Particularly, its application does not depend on the number of Eigenvalues. Theoretically, this approach can be applied for systems of degree > 3. But one has to bear in mind that stability checks difficulties may rise with the order of the system. Consequently, the rdm approach is limited for complex scenarios.

The paper proposes a strategy to set up a tractable model, where it is possible to discuss the stability of the resulting equilibria via the widely known Eigenvalue criterion. This seems more plausible, then to setup a very complex model right from the beginning without knowing ex ante if the resulting equilibria are stable or not. One has to note that by using this argument asymptotically stability cannot be shown. But this is not a restriction because the system can converge before time goes to infinity. In this paper we have shown by using a replicator dynamic model that the resulting equilibria for the case of constant, increasing and decreasing returns to scale are stable or saddle path stable, which is true for only one equilibrium. Thus, the focus of the is to expand the toolbox for discussing equilibria stability in replicator dynamics models, based on differential equations, notably for the case of more than one zero real Eigenvalues.

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