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Competition, Innovation and the Effect of Knowledge Accumulation

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Competition, Innovation and the Effect of Knowledge Accumulation*

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Abstract

The question this paper addresses is how the market structure evolves due to innovative activities when firms' level of technological competence is valuable for more than one project. The focus of the work is the analysis of the effect of learning-by-doing and organizational forgetting in R&D on firms' incentives to innovate. I develop a dynamic step by step innovation model with history dependency. Firms can accumulate knowledge by investing in R&D. As a benchmark I show that without knowledge accumulation the leader's R&D effort increases with the gap as she is trying to avoid competition in the future. When firms gain experience by performing R&D the resulting effect of knowledge induces technological leaders to rest on their laurels which allows followers to catch up. Contrary to the benchmark case, the leader's innovation effort declines with the lead. This causes an equilibrium where the incentives to innovate are highest when competition is most intense.

Keywords: competition, innovation, knowledge, market structure.

JEL classification numbers: L11, L13, O31, O41.

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1 Introduction

Innovation is an instrument for competitive advantage and often seen as one or even the engine for growth. Therefore it is crucial to understand its determinants. Competition and innovation are intimately connected. The relation is twofold. On the one hand incentives to innovate are driven by the competitive situation. On the other hand successful innovations affect and thus change the market structure. Due to this interdependency the impact of market structure on innovation can only be assessed if the converse direction – i.e. the changes in market structure caused by innovations – is accounted for. Hence, an analysis of the evolution of market structure due to innovations is best be done by means of a dynamic framework.

The link between product market competition and innovation has been studied for a long time. The classic contributions of Schumpeter and Arrow shaped the corresponding polar positions of competition hindering innovation (often attributed to Schumpeter) on the one hand and competition spurring innovation (often attributed to Arrow) on the other hand. Closely connected to the question whether incentives to innovate are increasing or decreasing with more intense market competition is the question on the endogenous evolution of the market. Putting aside changes in the number of firms (due to entry, exit or mergers and acquisitions), this reduces to the question on the evolution of differences between incumbent firms. Is one firm becoming more and more efficient leaving other firms behind or do we see neck and neck competition? Casual observations and empirical evidence suggest a process of action-reaction in markets, i.e. market leadership is constantly changing hands.[1] In theoretical analyses different modeling strategies lead to widely differing conclusions.

However, most of this literature seems to leave out some important aspects. It neglects that past experience in R&D usually has an impact on current success. Considering only one innovation project omits the fact that a level of technological competence may be valuable for following projects. In such a situation the innovation process never ends. One successfully completed project is supplanted by another project, sequentially or even simultaneously. On the one hand, the success in preceding projects helps in securing income. Beyond that, the pure experience of these projects improves performance in other projects. This is due to experience, learning-by-doing, user's feedback etc.

Our approach tries to identify the effect of experience in R&D in a stylized model designed to capture the essentials of the problem.

We develop a dynamic model with history dependency. History affects market opportunities, i.e. previous actions and outcomes determine the range of available actions and outcomes. This is modeled in a way that firm's investment in R&D does not only increase

[1] See Cohen and Levin (1989) for an early survey. More recently for example Khanna (1995), Lerner (1997) and Czarnitzki and Kraft (2004) find evidence for the fact that the firm that is behind engages in catch-up behavior, i.e. the status of being a challenger has a positive and significant impact while being a defensive firm has a negative impact on the incentives to innovate.

the chance of making a discovery, but furthermore increases the knowledge stock.[2] This knowledge stock is a measure of firm's past R&D effort and allows to model learning, i.e. firm's past experiences add to its current capabilities, and organizational forgetting. Learning-by-doing has been observed in many empirical works. In practice learning may occur when the innovation activities of a firm are adjusted due to past experiences or when innovation projects are cumulative, i.e. sequential and building on each other. An example would be an investment in laboratory equipment which could be used for other than the current project or gained experience of the researchers and developers. Organizational forgetting on the other hand is a phenomena that has been shown in more recent studies.[3] Sticking to the idea of knowledge capturing the experience of the workers, organizational forgetting would be the result of turnover and layoffs.

As described, firms' continuous investment in R&D creates the permanent possibility of a successful innovation. These innovations come in successive steps, i.e. a step has to be completed to proceed. Due to the "step-by-step" innovations a technological laggard must first catch up with the leading-edge technology before battling for technological leadership in the future. This in turn implies that if we do not see a process of increasing dominance then every once in a while competition will be neck-and-neck and therefore the escape competition effect will be strongest.[4] Regarding the product market, I assume the industry to be characterized by duopoly where firms are competing in prices. The incumbent firms simultaneously engage in R&D in order to decrease their relative costs.

The main research focus of the model where history and dynamics are essential is the effect of experience on the firms' incentives to invest in innovation activities. And how does this effect influence the evolution of market structure over time? What are the effects of competition in innovation on market structure? Does one firm become increasingly dominant by being more successful in R&D, i.e. do we see a process of increasing dominance, or is there a process of action reaction, in which market leadership is constantly changing hands? Above all we wish to discover when competition in innovation is most intense.

Starting with the benchmark case without learning we show that without the exogenous possibility of immediate imitation leader's R&D effort is increasing with the lead while laggard's effort is decreasing as the leader is trying to avoid competition in the future while the reduced prospect of moving ahead diminishes inventives for the follower. Nevertheless, leaders always invest less and hence a process of action reaction results.

Allowing for knowledge accumulation adds another effect. If one firm has accumulated enough knowledge its chances to successfully innovate are increased and therefore further R&D effort is less rewarding. The leading firm can afford to rest on its laurels and hence in steady state invests less the higher the technological lead. The knowledge effect outweighs

[2]This way of modeling is based upon the work of Doraszelski (2003).

[3]See for example Argote, Beckman, and Epple (1990) or Benkard (2000).

[4]This escape competition motive has been pointed out in previous theoretical work on innovation, for example by Mookherjee and Ray (1991).

the increased incentive for the leader to innovate in order to avoid competition. This may induce the follower to catch up.

With respect to product market competition our findings are in line with Arrow's position of competition spurring innovation. In our framework we clearly find that due to the effect of knowledge the incentives to perform R&D are increasing with the intensity of competition.

In addition to Sutton's work on industrial market structure (Sutton (1991, 1998, 2007)), the voluminous literature dealing with static models (See for example Belleflamme and Vergari (2006), Gilbert (2006) and Vives (2006)), this paper is especially related to the literature on dynamic evolution of oligopoly.

Budd, Harris, and Vickers (1993) present a work that analyzes whether the gap between two firms in a model of dynamic competition tends to increase or decrease. While modeling the gap in terms of an abstract (bounded) state of competition parameter without modeling the product market explicitly they find that the gap tends to evolve into the direction where joint payoffs are greater. This most often results in a process of increasing dominance. Cabral and Riordan (1994) provide further indications of increasing dominance. Segal and Whinston (2005) study the effects of antitrust in a dynamic R&D model based on "winner-take-all" competition. Ericson and Pakes (1995) develop a comprehensive model of industry behavior with firm specific sources of uncertainty. The work is more considered as a model for industry dynamics due to entry, exit and mergers. Besides, as this model is highly complex the authors have to use numerical methods.

Papers analyzing industry evolution when there is learning-by-doing like Dasgupta and Stiglitz (1988) and Cabral and Riordan (1994, 1997) usually simply model cost reduction as a function of output decisions. Basically, firms learn by producing not by researching and developing. That means it becomes less costly for the leader to gain higher profits as the lead widens. With this way of modeling R&D is complementary with production. Besides, organizationally forgetting can not be modeled in these frameworks.

Our work is also related to the literature on patent races (See Reinganum (1989) for an early summery). Due to the endpoint that players are aiming for, usually the property of increasing dominance results. This characteristic remains in multistage race models, where several stages are introduced into a patent race, as there is still a definite end.[5] To the best of our knowledge Doraszelski (2003) was the first introducing knowledge accumulation into patent races. However, he does not model product market competition.

Regarding dynamic step-by-step innovation, our work is most closely related and extends the works of Aghion, Harris, Howitt, and Vickers (2001) and Acemoglu and Akcigit (2006). Although our model builds on these papers, it also differs from them in significant ways. Most importantly, our main research question regards the

[5] See for example Fudenberg, Gilbert, Stiglitz, and Tirole (1983), Harris and Vickers (1985), Grossman and Shapiro (1987), Harris and Vickers (1987) and Lippman and McCardle (1988).

effect of experience in R&D. Therefore we extend the model to learning-by-doing and organizational forgetting. Besides, we do not imply the strong assumptions on imitation as Aghion et al. (2001) and Acemoglu and Akcigit (2006). These authors assume the follower at least catches up with the frontier technology with one successful innovation.[6] Acemoglu and Akcigit (2006) show numerically, based on the model of Aghion et al. (2001), that optimal intellectual property rights policy provides more protection to firms that are technologically more advanced as this policy strengthens the escape competition effect. Obviously, R&D by firms that are sufficiently ahead is encouraged just as well as effort by companies with a limited lead because of their prospect of reaching levels of gaps associated with higher protection. That is to say the effect of avoiding competition that is absent in the basic model is introduced by means of intellectual property rights policy.

Based on the work of Aghion et al. (2001), Aghion, Bloom, Blundell, Griffith, and Howitt (2005) analyze the relationship between product market competition and innovation. However, they only allow for two possible states (one step behind and neck-to-neck). In a related work Hörner (2004) develops a model allowing a firm to be an arbitrary number of steps ahead or behind. His contribution and the effect of a firm being sufficiently far ahead suggests that a analysis à la Aghion et al. (2005) with only two possible states leaves out some aspects. Unfortunately, Hörner does not model product market competition.

Our work differs from all of the above papers in that we consider the effects of learning-by-doing and organizational forgetting in R&D with firms competing on the product market.

The rest of the paper is organized as follows. Section 2 presents the basic model. Section 3 provides an analysis of optimal R&D when accumulation of knowledge is not possible. In this section we also compare our result to the one of the related framework of Aghion et al. (2001). Section 4 analyzes the equilibrium R&D investment when the effect of knowledge is at place and compares it with the benchmark case without knowledge. Section 5 concludes while the Appendix contains the proofs of the results stated in the text.

2 The Model

We consider an industry with two ex-ante symmetric firms $i = 1, 2$ producing homogeneous goods.[7] Firms' costs of production depend on their technologies. A firm's technol-

[6] Acemoglu and Akcigit (2006) also consider the case where the follower might even be able to improve over the frontier technology.

[7] Extending the derived results to the more general case of differentiated goods would be possible at the cost of additional notation and a considerably higher complexity in derivation. As only minor additional insights can be gained by such an extension as long as the degree of substitution is exogenous we restrict attention to the case of perfect substitutes.

ogy is given by x_i, and a firm produces output quantity y at cost $c_i(y) = ye^{-x_i}$. Each firm can continuously engage in R&D in order to improve its technology and thereby decrease its relative cost. Innovative investment is denoted by z_i. Innovations are uncertain and come in successive steps. Hence, a step has to be completed to proceed. When a firm moves one technical step ahead its technology increases by one.[8]

Investments in R&D increase the chance of a successful innovation, i.e. the chance of moving one step ahead. Besides, there is another effect of R&D. Firms accumulate knowledge. A firm's gathered knowledge is denoted by k_i and evolves according to

$$\frac{dk_i}{dt} = \dot{k}_i = u(z_i) - \delta k_i. \tag{1}$$

Here, u_i is firm i's rate of knowledge acquisition. We assume the rate of knowledge acquisition to be a function of investment in R&D given by $u_i = u(z_i) = (\eta z_i)^{\frac{1}{\eta}}$ so that the cost incurring to acquire knowledge at rate u_i is $z(u_i) = \frac{1}{\eta} u_i^{\eta}$ and $\eta > 1$ measures the elasticity of the cost function. Hence, the R&D-cost function is an increasing and convex function. The depreciation rate of the knowledge stock is given by $\delta \geq 0$.

The more knowledge a firm has accumulated, the more successful – in expectation – is the firm's R&D. Hence, the distribution of a firm's success times, given by the firm's hazard rate h_i, does not only depend on the current investment z_i but also on past effort measured by the knowledge stock k_i.[9] A firm moves one technical step ahead with hazard rate

$$h_i = \lambda u(z_i) + \gamma k_i^{\alpha}. \tag{2}$$

A firm's hazard rate of successful innovation is the rate at which the discovery is made at a certain point in time given that it has not been made before. The parameter λ measures the effectiveness of current effort while γ measures the effectiveness of past effort. The marginal impact of past R&D efforts is determined by α. A firm's technology follows a Poisson process $dx_i(t) = 1 \cdot dq_i(t)$ where $q_i(t)$ is the underlying process with the non-constant hazard rate $h_i(t)$.[10]

If $\gamma > 0$, the model allows for history dependency. Hence, R&D effort for one project is – by means of the gathered knowledge – valuable for the following projects. This allows to model learning and organizational forgetting. In general learning means a firm's past experiences add to its current capabilities. Organizational forgetting is modeled as depreciation of knowledge. This implies that a firm's recent experiences are more important and valuable than older know-how. Organizational forgetting is captured in the model by setting $\delta > 0$.

[8]The stepsize is arbitrarily set equal to one. As long as the size is exogenous and constant all results remain unchanged with a different increment. However, allowing for different sizes of innovations may alter the outcome considerably.

[9]Note that due to this way of modeling we cannot interpret knowledge as capital in the usual way since knowledge is not an input factor in production and knowledge as such does not influence the production technology in a direct way.

[10]For detailed information on Poisson processes see Ross (2003).

Firms are assumed to be Bertrand competitors and maximize expected discounted profits. Demand at price p is given by $y(p) = \frac{1}{p}$. The instantaneous profit in Bertrand equilibrium then only depends on the technology gap leaving the laggard j with nothing while the leader i earns $1 - e^{-x_i(t)+x_j(t)}$.[11] With the technology gap $\Delta_i(t) \equiv x_i(t) - x_j(t)$ instantaneous profits are

$$\pi_i(\Delta_i, t) = \begin{cases} 1 - e^{-\Delta_i(t)} & \text{for } \Delta_i(t) > 0, \\ 0 & \text{for } \Delta_i(t) \leq 0. \end{cases}$$

On top of these profits both firms have to pay their investment z_i in R&D. Note that even if the industry is leveled, i.e. $\Delta(t) \equiv |\Delta_i(t)| = |\Delta_j(t)| = 0$, the situation is not necessarily symmetric since firms may (and most often will) dispose of different knowledge stocks.

Figure 1 shows how the firm's market profit varies with the size of the lead Δ. It shows that profit increases slower the higher the lead already is, i.e. $\frac{\partial \pi_i(\cdot)}{\partial \Delta_i} > 0$ and $\frac{\partial^2 \pi_i(\cdot)}{\partial \Delta_i^2} < 0$ for $\Delta_i > 0$. Thus, the motive of escape competition is potentially more important for firms in the neck-and-neck state. On the other hand in an industry with a large technological gap neither firm makes much immediate gain from innovating; the leader is already earning almost the maximum possible profit and the follower will still earns nothing even if he catches up.

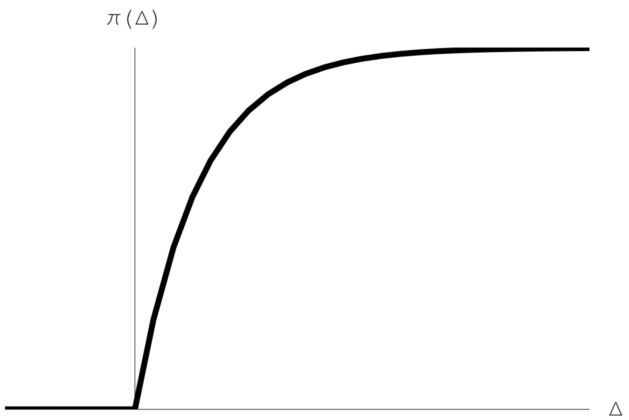

Figure 1: A firm's profit π as a function of its technological lead Δ.

Firms are assumed to maximize expected discounted profits with time preference rate $\rho \in [0, 1)$. As firm i's instantaneous profit is $\pi_i(\Delta_i, t)$, the firm's objective function to be maximized over z_i is

$$\Pi_i(t) \equiv E_t \int_t^\infty (\pi_i(\Delta_i, \tau) - z_i(\tau))\, e^{-\rho(\tau - t)} d\tau. \tag{3}$$

[11]For the sake of readability throughout the rest of the paper we will denote firm i's competitor by j, i.e. $j \neq i$ will always hold. Besides, we suppress the indication of time where not necessary.

We next analyze the equilibrium research intensities. We assume that these equilibrium innovation rates are determined by the necessary conditions for a Markov-stationary equilibrium (steady state fraction of states) in which each firm seeks to maximize expected discounted profits. Hence, firm i maximizes its objective function (3) subject to the evolution of the knowledge stocks (1) and the technologies (2).

3 Equilibrium without Acquisition of Knowledge

As a benchmark and starting point we analyze the extreme case, when there is no effect of knowledge and only current effort counts. In this case the state variables are x_1 and x_2. Due to the modeling approach we can use $\Delta_i \equiv x_i - x_j$ as the only state variable. As we do not have to distinguish the impact of past and current effort we can set λ arbitrarily equal to one.

With this exclusion of knowledge our benchmark model is very similar to the one of Aghion et al. (2001), but there is one crucial difference. Aghion et al. (2001) and also Acemoglu and Akcigit (2006) assume that the laggard can always catch up with the industry's leader with only one successful innovation no matter how big the gap is. That means the R&D cost function of catching up is independent of the gap that has to be bridged. Therefore, there is no strategic motive for performing R&D. The only incentive for leaders to increase the industry's gap results from the immediate increase in profit. On the long run being sufficiently far ahead does not provide any competitive advantage in future R&D. The converse is true for the follower, i.e. being far behind is not disadvantageous for future competition. In fact, for followers the current gap is irrelevant as it even does not influence product market competition.

We contrary assume the laggard has to catch up step by step.[12] Hence, being sufficiently far ahead provides advantages in future technological competition and strategic effects are at place to invest in R&D. This is the case for the laggard and the leader.

Besides, in the framework of Aghion et al. (2001), their assumption on imitation is expected to have similar effects as knowledge accumulation, namely reducing innovation incentives for leaders. When the follower has no possibility of imitating the leader's technology we are able to disentangle the knowledge effect in the next section.

3.1 Optimal R&D Effort

In this section we analyze some properties of the firms' optimal effort in R&D. For the sake of simplification we assume firms maximize over u instead of z. To solve for the Markov-stationary equilibrium we use dynamic programming methods. This yields the

[12] A discussion on how realistic these assumptions are is given at the end of this section.

maximized Bellman equations:

$$\rho V_i(\Delta_i) = \pi_i(\Delta_i) - z(u_i) + [V_i(\Delta_i + 1) - V_i(\Delta_i)]\, h_i(u_i(\Delta_i)) + [V_i(\Delta_i - 1) - V_i(s)]\, h_j(u_j(\Delta_i)). \tag{4}$$

These equations state that the annuity value $\rho V_i(\Delta_i)$ of each firm i in industry state Δ_i at any date t equals the current profit flow $\pi_i(\Delta_i) - z(u_i)$ plus the expected capital gain $[V_i(\Delta_i + 1) - V_i(\Delta_i)]\, h_i(u_i(\Delta_i))$ from moving one technological step forward plus the expected capital loss $[V_i(\Delta_i - 1) - V_i(\Delta_i)]\, h_j(u_j(\Delta_i))$ from having the competitor stepping forward.

With $\lambda = 1$ and $\eta = 2$ we get the following relations of optimal R&D effort:[13]

Lemma 1. *Assuming $\eta = 2$, the optimal R&D effort satisfies the following equations:*

1. *When firms are neck-and-neck, i.e. $\Delta = 0$:*

$$u(0) = \sqrt{2 - \tfrac{2}{e} + \rho^2 + u(1)^2} - \rho; \tag{5}$$

2. *For the industry's leader with $\Delta > 0$:*

$$u(\Delta) = u(-\Delta - 1) - \rho + \sqrt{e^{-\Delta}\left(2 - \tfrac{2}{e}\right) + (u(-\Delta - 1) - \rho)^2 - 2u(\Delta - 1)u(-\Delta) + u(\Delta + 1)^2}; \tag{6}$$

3. *For the follower, i.e. $-\Delta < 0$:*

$$u(-\Delta) = \tfrac{1}{2}u(\Delta - 1) - \tfrac{\rho}{2} + \sqrt{\tfrac{1}{4}\left(u(\Delta - 1) - \rho\right)^2 - u(-\Delta - 1)u(\Delta) + u(-\Delta + 1)^2}. \tag{7}$$

Proof. See Appendix A.1.

From Lemma 1 we cannot find a closed form solution for optimal R&D effort as a function of the gap but under the assumption of $\rho = 0$[14] we can derive a pattern regarding the optimal R&D investment:

Proposition 2. *Assuming $\eta = 2$ and a time preference rate $\rho = 0$, firms' optimal behavior satisfies the following conditions:*

- *R&D investment is highest for a firm being one step behind and the effort of the laggard decreases with the gap, i.e. $z(-1) > z(-2) > z(-3) \ldots$.*

[13] For the sake of readability we will suppress the identity of the firm where not necessary.

[14] We were able to show numerically that the results basically hold with $\rho > 0$ in quality (with the additional feature that R&D investment eventually falls to zero). However, the analytical derivation is excessively more complex.

- *Investment of a laggard is always higher than that of a neck and neck firm, i.e.* $\forall \Delta > 0 : z(-\Delta) > z(0)$.
- *Investment of a leader increases with the gap, i.e.* $z(1) < z(2) < z(3) < \dots$.
- *Investment of a leader is always smaller than that of a neck-and-neck firm, i.e.* $\forall \Delta > 0 : z(\Delta) < z(0)$.

Proof. See Appendix A.2.

The pattern resulting from the statements of proposition 2 is illustrated in figure 2.

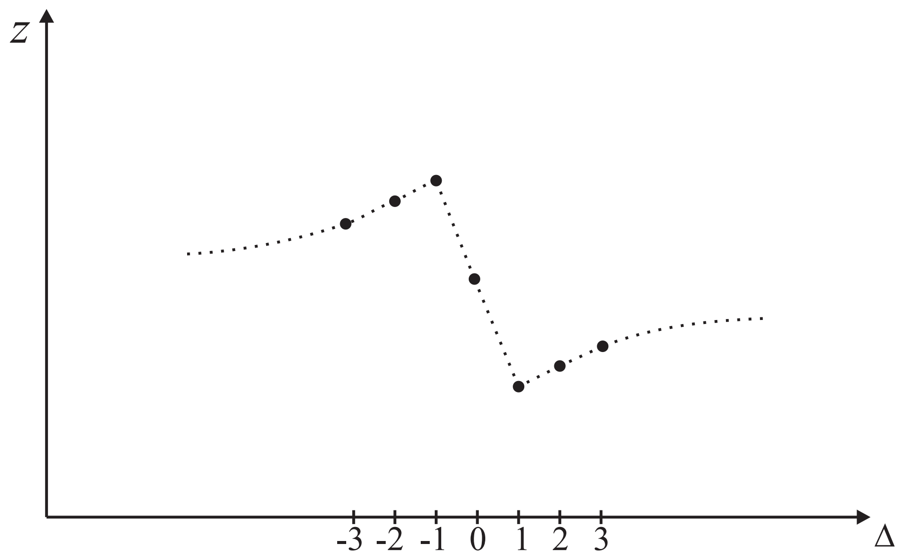

Figure 2: Optimal R&D effort subject to the firm's gap.

We see that the R&D effort of a firm being exactly one step behind provides the highest incentive to perform R&D. When the laggard falls further behind, the usual Schumpeterian effect of a reduced prospect of moving ahead diminishes incentives.

The opposite is true for the leader. The incentive is lowest when being one step ahead and increases when moving further ahead. The motive for this increasing effort is not the raise in immediate profit but the raise in expected future profit. When the leader moves ahead she decreases the probability that the laggard catches up within a certain time and hence she increases the expected duration of maintaining positive profits.

The greatest R&D effort to enhance the leading edge technology is made when both companies dispose of this technology, i.e. in a neck-and-neck industry. This result is due to the escape-competition effect. It is clear that neck-and-neck firms innovate to escape the strong competition on the product market.

Interestingly, our result quite differs from the result in Aghion et al. (2001). In their model the leader's effort decreases while the follower's effort increases with the industry's gap. This difference in outcome is due to the mentioned difference in modeling: Aghion et al. (2001) assume that the R&D cost function of catching up is independent of the technological gap to be made up. Due to this strong assumption the only incentive for the industry's leader to innovate is a further increase in profit while in our model strategic effects are at place. By widen the technological gap the industry's leader makes it more difficult for the follower to catch up. As a result, we see two characteristics of the escape-competition effect. On the one hand firms in a neck-and-neck industry perform R&D to escape the competition while firms that are technologically advanced innovate to avoid competition in the future. In this framework without imitation we can subdivide the escape competition effect into the basic effect at work in a neck-and-neck state and the "avoid-competition effect" at work in a staggered industry.[15]

A similar incentive scheme holds for the follower. In the model à la Aghion et al. (2001) the follower can always catch up immediately and battle for industry leadership. Therefore, the described Schumpeterian effect is almost absent. Here in contrast, a follower being sufficiently far behind has to invest a large amount into R&D to get into the position of being able to battle for market leadership. Hence, the incentive to invest is decreasing with the gap for the follower and increasing for the leader and the described incentive scheme seems to be exactly opposite to Aghion et al. (2001).

But this is not the whole story. In fact, our model does to some extent incorporate the model of Aghion et al. (2001). As in their model there is no strategic effect of competition in innovation, our states $\Delta_i \in \{-1, 0, 1\}$ contain the basic features of their framework. Considering only these states leaves out the effect of changes in the leader's profit but can still be used to show the basic result. With a gap not bigger than one, the leader can catch up immediately as it is the case in the model of Aghion et al. (2001). Taking only these states into account, we find the same result, namely a decrease in R&D with the gap. Strategic effects come into place when the lead widens.

Thus, we can conclude that the incentive scheme resulting in Aghion et al. (2001) is mainly a result of the strong imitation assumption which even Aghion et al. consider as not very realistic and a point for extension. Obviously, our extreme case is not the most realistic scenario either as this would be in between the two extreme cases. However, we were able to show the additional effects when strategic motives to perform R&D come into place. The outcome of a more realistic framework where imitation is possible to some extent would be in between the two extreme case results, depending on how catching up with the leading edge technology is possible. This is the realm of intellectual property rights policy.

[15]It is exactly the strengthening of this avoid competition effect that drives the results of Acemoglu and Akcigit (2006).

3.2 Steady State Industry Structure

With the results derived so far we will now analyze the industry's structure in steady state. As the leader's R&D effort is always smaller than the laggard's effort the firms will not drift apart in expectation and a steady state exists.[16]

Let μ_Δ denote the steady-state probability of the industry showing a technological gap Δ. As we do not consider knowledge acquisition, a firm's effort $u(\Delta)$ equals the transition rate. Stationarity implies that for any state Δ the flow of industries into this state Δ must equal the flow out. Consider first state 0 (neck-and-neck). During time interval dt, in $\mu_1 u(-1)dt$ in industries with technological gap 1 the follower catches up with the leader, hence, the total flow of industries into state 0 is $\mu_1 u(-1)dt$. On the other hand, in $\mu_0 \cdot 2u(0)dt$ in neck-and-neck industries one of the two firms acquires a lead, hence the total flow of industries out of state 0 is $2\mu_0 u(0)dt$. Thus in steady state

$$2\mu_0 u(0) = \mu_1 u(-1).$$

Replicating the same reasoning for all states yields

$$\mu_1(u(1) + u(-1)) = 2\mu_0 u(0) + \mu_2 u(-2),$$
$$\mu_2(u(2) + u(-2)) = \mu_1 u(1) + \mu_3 u(-3),$$

and in gerneral

$$\mu_\Delta(u(\Delta) + u(-\Delta)) = \mu_{\Delta-1} u(\Delta - 1) + \mu_{\Delta+1} u(-\Delta - 1) \text{ for all } \Delta > 1. \tag{8}$$

Using these conditions, it is easy to see, that

$$\mu_\Delta u(\Delta) = \mu_{\Delta+1} u(-\Delta - 1) \text{ for all } \Delta \geq 1 \tag{9}$$

has to hold.

With the derived stationary conditions it is possible to determine the steady state growth rate. The growth rate of the industry is asymptotically given as $g = \lim_{\Delta t \to \infty} \frac{\Delta \ln y}{\Delta t}$ with y as industry's output.[17]

The quantity sold by the industry as a whole grows at rate e with every step the follower catches up. Thus, over any long time interval, the logarithmic change in output can be approximated by the number of times the follower catches up one step over the time interval. The asymptotic frequency of a catch up equals the steady state flow of the industry from state Δ to state $\Delta - 1$, which in turn equals the fraction μ_Δ of industries

[16] See Acemoglu and Akcigit (2006) for a formal proof on the existence of a steady state.

[17] Although we do not model an entire closed economy and cannot provide a general equilibrium analysis, our model can easily be transferred into such a framework. Thus, we can draw conclusions on the economy's growth rate from the growth rate of the industry or sector. In a general equilibrium framework with an economy consisting of a mass of 1 identical industries, the defined industry growth rate g equals the growth rate of the economy $\frac{d \ln Y}{dt}$ with Y as the economy's aggregate output.

in state Δ times the transition rate that the follower catches up one step. This is given by $u(-\Delta)$. Hence, $g = \sum_{\Delta \geq 1} \mu_\Delta u(-\Delta)$ which using the stationary conditions (9) can be written as

$$g = 2\mu_0 u(0) + \sum_{\Delta \geq 1} \mu_\Delta u(\Delta). \tag{10}$$

Equation (10) states the following proposition:

Proposition 3. *The steady state growth rate in a step-by-step innovation model equals the frequency of frontier innovation, i.e. innovations by industry leaders and neck-and-neck firms, which advance the industry's frontier technology.*

This proposition shows how neck-and-neck rivalry promotes growth. When an industry is neck-and-neck there are two firms trying to advance the industry's frontier technology, whereas in all other states just one firm is trying. Thus, even if all the efforts were the same, technology would advance in average twice as fast in neck-and-neck industries as in any other.

Moreover, as we have seen the R&D effort of a neck-and-neck firm is always greater than that of a leader, such an industry grows more than twice as fast as other industries. Note that the described characteristic of the steady state growth rate is not a consequence of the no-knowledge assumption but rather the result of any similar step-by-step model showing a steady state.

4 The Effect of Knowledge

Now, we analyze the situation when knowledge is introduced, i.e. the market is modeled as described in section 2. The industry's state will be denoted as $s \equiv (\Delta_1, k_1, k_2)$.

Using dynamic programming methods for the problem given by (3) subject to (1) and (2) yields the maximized Bellman equations for the firms:

$$\begin{aligned} \rho V_i(s) =& \pi_i(\Delta_i) - z_i(s) + \left[V_i(x_i+1,\cdot) - V_i(s)\right] h_i(z_i(s)) + \left[V_i(x_j+1,\cdot) - V_i(s)\right] h_j(z_j(s)) \\ &+ \frac{\partial V_i(s)}{\partial k_i}\left(u(z_i(s)) - \delta k_i\right) + \frac{\partial V_i(s)}{\partial k_j}\left(u(z_j(s)) - \delta k_j\right). \end{aligned} \tag{11}$$

Again, the annuity value $\rho V_i(s)$ of firm i in industry state s at date t equals the current profit flow $\pi_i(\Delta_i) - z_i(s)$ plus the expected capital gain $\left[V_i(x_i+1,\cdot) - V_i(s)\right] h_i(z_i(s))$ from moving one technological step forward plus the expected capital loss $\left[V_i(x_j+1,\cdot) - V_i(s)\right] h_j(z_j(s))$ from having the competitor stepping forward. Now, two other terms are added, namely the capital gain from increased knowledge $\frac{\partial V_i(s)}{\partial k_i}\left(u(z_i(s)) - \delta k_i\right)$ and the capital loss $\frac{\partial V_i(s)}{\partial k_j}\left(u(z_j(s)) - \delta k_j\right)$ from the competitor's acquired knowledge.

To apply the dynamic programming methods we make another simplifying assumption, namely that investment in R&D does not immediately influence a firm's probability of success. Hence, the parameter λ is assumed to be zero. This yields the following proposition:

Proposition 4. *When investment in R&D has no immediate influence on the chances of a successful innovation, i.e. when $\lambda = 0$, firm's optimal investment does not immediately react when a jump in the firm's own or the competitor's technology occurs.*

Proof. See Appendix B.

This result is not very surprising and a direct consequence of the assumption of $\lambda = 0$. Since firms cannot react directly on technology jumps they don't and hence investment in R&D does not jump when technology does.

And there is another consequence of assuming $\lambda = 0$. A steady state fails to exist. A stationary Markov chain would imply that for any state s the flow of industries into state s must equal the flow out. This again implies hazard rates and hence knowledge stock to immediately react on technological jumps which cannot be the case in this framework. However, we can still determine the optimal rule describing the evolution of investment under firms' optimal behavior. Using the result of proposition 4 in the dynamic programming approach yields:

Lemma 5. *When investment in R&D has no immediate influence on the chances of a successful innovation, i.e. when $\lambda = 0$, optimal investment evolves according to*

$$\frac{dz_i(s)}{dt} = z_i(s)\frac{\eta}{\eta-1}\Big(\rho + \delta - (\eta z_i(s))^{\frac{1-\eta}{\eta}}\Phi(\Delta_i)\alpha\gamma k_i^{\alpha-1}\Big), \tag{12}$$

with $\Phi(\Delta) > 0$, $\frac{\partial\Phi(\Delta)}{\partial\Delta} < 0$ for $\Delta > 0$ and $\frac{\partial\Phi(\Delta)}{\partial\Delta} > 0$ for $\Delta < 0$.

Proof. See Appendix B.

We can immediately and clearly see from (12) what the direct effect of knowledge is: The more knowledge a firm has acquired the smaller is the growth rate of optimal investment. This illustrates the mentioned effect of resting on its laurels. Firms acquire knowledge by investing in R&D. Hence, the knowledge stock grows and the more it grows the less do firms invest since they can afford to be based on this stock. Although knowledge as such does not enter in the production function, knowledge is productive in terms of expectations and therefore valuable for firms.

The effect of knowledge in the long run is more difficult to assess. As $z_i(s)$ on the right hand side depends on a firm's own and the competitors knowledge, equation (12) does not immediately tell the long run effects of knowledge on the evolution of the market.[18]

Besides, there is another effect of investment, namely technological progress. R&D effort induces firms to technological move ahead. Thus, by investing in R&D firms

[18] Note that different to technology levels there is no direct effect of the competitor's knowledge on optimal investment.

do not only accumulate knowledge but in expectation also increase their technology. We know that $\frac{\partial\Phi(\Delta)}{\partial\Delta} < 0$ for $\Delta > 0$. Therefore, from (12) it is clear that investment grows faster the higher the technological lead. For the follower we know that $\frac{\partial\Phi(\Delta)}{\partial\Delta} > 0$ for $\Delta < 0$. Hence, the firm that is behind invests more and more the closer it gets. This shows again the effects described for the benchmark case (cf. proposition 2).

To see how these effects influence each other in the long run we would need to assess the overall dynamic properties of the model in terms of steady states. Unfortunately, as Δ follows a stochastic process a steady state does not exist. However, we can for the moment assume Δ to be constant to get an idea of the dynamics. Using equation (12) we are able to analyze "temporary steady states". This approach is closely related to literature on natural volatility. This relatively new, mainly macroeconomic approach jointly analyzes short-run instability and long-run growth due to innovations.[19] Economies fluctuate because of some mechanism that results from decisions of agents. In this literature mostly a determinant for production technology like total factor productivity does not grow smoothly over time but follows a step function. In stochastic natural volatility models the probability with which a shock occurs depends on the decision on agents, i.e. it is endogenous. In our model we make use of the idea of temporary steady states.

From (12) it is clear that besides the trivial temporary steady state $z = 0$ and $k = 0$, there is the locus $\frac{dz}{dt} = 0$ at

$$z = \frac{1}{\eta}\left(\frac{\delta+\rho}{\alpha\gamma\Phi(\Delta)}\right)^{\frac{\eta}{1-\eta}} k^{\frac{(1-\alpha)\eta}{1-\eta}}. \tag{13}$$

From (1) we have the locus $\frac{dk}{dt} = \dot{k} = u(z) - \delta k = 0$. These two loci partition the space $\{k, z\}$ into different regions. Temporary steady states are identified by intersections between loci. The properties of steady states depend on the shape of the locus $\frac{dz}{dt} = 0$ and this again on the marginal impact of knowledge determined by α and the elasticity of the cost function η. In either case the loci partition the space $\{k, z\}$ into four regions. We obtain one intersection of loci (1) and (13) for positive values of z and k and hence one nontrivial steady state point P. The situation is illustrated in figure 3. The graph on the left shows the phase diagram for the hazard rate being a concave function of knowledge ($\alpha < 1$) while the right diagram shows the case where the hazard rate is a convex function of knowledge ($\eta > \alpha > 1$). The dynamics are summarized by vertical and horizontal arrows.

For $\alpha < 1$ the function described by (13) is decreasing for all $k > 0$ while the function given by $\dot{k} = 0$ has a positive and increasing slope. Therefore, we obviously obtain one intersection and hence one nontrivial steady state point P.

[19]Important papers in this strand of literature include for example Bental and Peled (1996), Matsuyama (1999), Francois and Lloyd-Ellis (2003), Maliar and Maliar (2004), Gabaix (2005) and Haruyama (2005). In these macroeconomic models the motivation for fluctuations in aggregate growth and the link to long run growth are important issues that are irrelevant in our model.

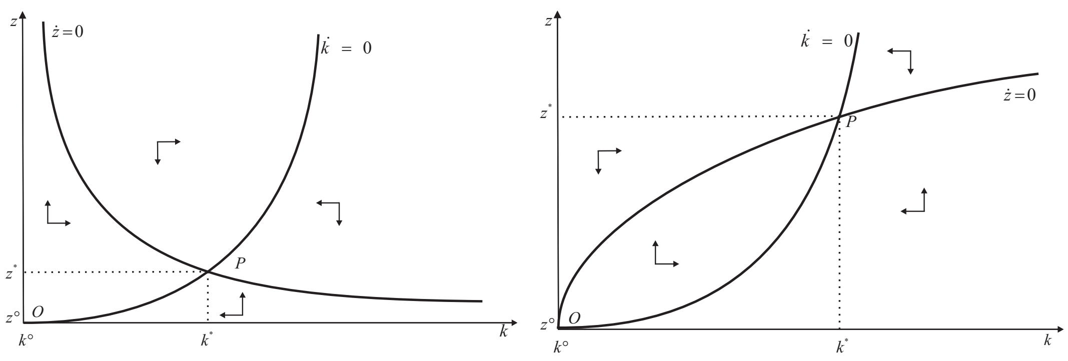

Figure 3: Convergence to the temporary steady state P for $\alpha < 1$ and $\eta > \alpha > 1$.

When $\eta > \alpha > 1$ the function given by (13) is increasing with a decreasing slope while the function given by $\dot{k} = 0$ has an increasing slope.[20] Furthermore it is easy to show that $\dot{z} = 0$ is steeper for sufficiently small values of k. Therefore, we obtain two intersections and hence two steady state points O and P, where O is again the trivial steady state. For $\alpha > \eta$ both functions are increasing with an increasing slope but different second derivatives with respect to k. Hence, again two intersections corresponding to O and P result. Only for the special case of $\alpha = \eta$ just the trivial steady state exists.[21]

It is easy to see that in all cases (except from $\alpha > \eta$ and $\left(\frac{\delta+\rho}{\alpha\gamma\Phi(\Delta)}\right)^{\frac{\eta}{1-\eta}} > \delta^{\eta}$) points converge towards the stationary equilibrium P at (k^*, z^*). Hence, the equilibrium P is always reached and stable as long as Δ does not jump. Note that the described results hold for both firms in the economy simultaneously, i.e. as long as Δ does not jump, knowledge and R&D investment for leader and follower converge to 2 different steady states.

When one firm successfully implements an innovation Δ jumps. When the leader innovates the technological gap increases. This causes the line $\dot{z} = 0$ for the leader to decrease while for the follower it increases. Hence, the new temporary steady state towards points converge is left and below the old one for the leader and right and above for the follower. The converse is true when the follower innovates. These dynamics are illustrated for $\eta > \alpha > 1$ in figure 4. The new loci are shown by the dashed lines. We see that steady state investment and knowledge both decrease for the leader when being successful while in opposition these do increase for the follower.

Subsequently, the economy approaches towards the new steady states until another jump in technology occurs which might move firms towards a former steady state again. These cyclical equilibria are described by a "Sisyphus-type" behavior. Investment and

[20]The hazard rate being a linear ($\alpha = 1$) function of knowledge is a special case where the function described by (13) is a horizontal line. The results are similar to the described cases and therefore not given in detail.

[21]For the very special case of $\alpha = \eta$ and $\left(\frac{\delta+\rho}{\alpha\gamma\Phi(\Delta)}\right)^{\frac{\eta}{1-\eta}} = \delta^{\eta}$ the two functions are identical and we have an infinite number of steady states.

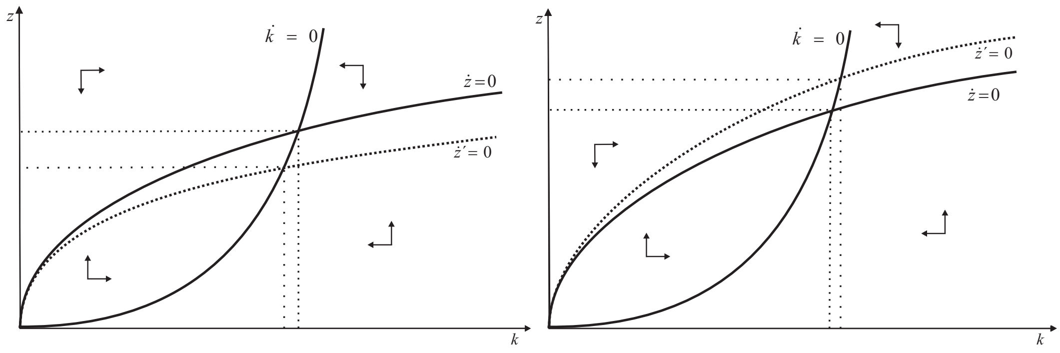

Figure 4: Temporary steady states before and after a successful innovation by the leader when $\eta > \alpha > 1$. The leader's fluctuation can be seen on the left, the follower's on the right phase diagram.

knowledge approach the steady state but are thrown back due to the implementation of a new successful innovation.

The resulting investment z^* in the temporary steady state P is given as

$$z^* = \frac{1}{\eta}\left(\frac{\delta+\rho}{\alpha\gamma\Phi(\Delta)}\right)^{\frac{\eta}{\alpha-\eta}} \delta^{\frac{(\alpha-1)\eta}{\alpha-\eta}}. \tag{14}$$

We now proceed to the comparative statics on z^* w.r.t. the technological gap. As long as $\alpha < \eta$ it is verified from (14) that for the leading firm

$$\frac{\partial z^*}{\partial \Delta} < 0$$

holds true. The result is opposite for the follower, i.e. when $\Delta < 0$:

$$\frac{\partial z^*}{\partial \Delta} > 0.$$

This can be summarized by the following proposition.

Proposition 6. *As long as $\alpha < \eta$, the leader's and follower's temporary steady state investment in R&D is decreasing with the technological gap.*

In contrast to the benchmark case, now ceteris paribus steady state investment for the leader is decreasing with the technological lead. She has accumulated enough knowledge such that her chances to successfully innovate and to maintain positive profits are sufficiently high and further R&D effort is less rewarding.

On the other hand, the follower invests more the closer he gets. His incentive scheme is basically the same as in the benchmark case, i.e. the reduced prospect of moving ahead diminishes incentives to innovate when the gap increases.

Comparing the result here with that of the benchmark case we can clearly identify the effect of knowledge: As investment in R&D is never lost the leading firm can afford to scale back its R&D effort. These results might cause the described process of action reaction. The leader rests on her laurels which allows the follower to catch up. The result is a market where leadership is constantly changing hands.

Obviously, struggle is fiercest when firms are shoulder to shoulder and the intensity of competition is higher the closer the technologies of the firms. Thus, the incentive to invest in innovation is increasing with the intensity of competition. This result can be interpreted in the light of the debate between the polar positions attributed to Schumpeter and Arrow, concerning the relationship between the intensity of market competition and the incentives to invest in R&D. Here, the position of Arrow of competition spurring innovation finds support.

5 Conclusion

In this paper, we developed a dynamic step-by-step innovation framework where firms' level of innovative competence is valuable for more than one R&D project to investigate the impact of knowledge on firms' optimal innovative effort and the evolution of industrial market structure.

The focus has been the general questions of whether the firm that is currently in the lead tends to increase its advantage over its rival, or whether there is a tendency for the rival to catch-up. We attempted to determine the effect of learning-by-doing and organizational forgetting in R&D on firms' incentives to innovate.

To address these questions we analyzed a model where the state of competition is represented in one dimension. In the model firms engage in step-by-step innovation. Leaders can innovate in order to widen the technological gap between themselves and the follower. This does not only increase their profit but also decreases the probability of getting caught up by the follower. The follower on the other hand innovates to first catch up step by step with and then to surpass the leader. Firms acquire knowledge by engaging in R&D projects. This knowledge is valuable not only for the current but also for future projects. Hence, successful projects provide a competitive advantage on the product market and in innovation activities.

To be able to assess the effect of knowledge, we first analyzed the case where knowledge is worthless for R&D. As the possibility of imitation for the follower as well as the effect of knowledge accumulation induce the leader to invest less in R&D the higher the gap, we exclude the possibility of imitation so to disentangle these two effects. Besides, the exclusion of imitation adds strategic motives to competition in innovation. We found that a leader in an economy without the possibility of imitation increases her innovative effort the further away she moves as she is trying to avoid competition in the future.

Introducing the possibility of gaining experience by innovative activities adds the knowledge effect which outweighs the avoid competition effect and the leader's R&D effort decreases with the lead. She rests on her laurels which in turn might induce the follower to catch up. Besides, we see that when knowledge is at place the incentives to innovate are higher the higher the intensity of competition. Hence, competition spurs innovation.

The main aim of the paper has been to understand the incentives generated by learning-by-doing and organizational forgetting and how these incentives influence the evolution of the market.

Nevertheless, these findings are based on two extreme cases of a rather simple analytical model. It would be interesting to see whether the results of the model in general are in line with our special case results. This could be done by means of a numerical analysis. Intuitively, one would assume that the result of such a more general analysis would be a mixture of the two given scenarios. Depending on the parameters determining the impact of past and current R&D effort, the result would either go more into the direction of the benchmark case of section 3 or the pure knowledge case of section 4.

Furthermore, investigating the impact of intellectual property rights policy could be a revealing task. This could even be done without considering knowledge. On the one hand, a less protective policy would make catching up easier and the industry would more often show a neck-and-neck state in which the growth rate is highest. On the other hand, such a policy would diminish the Schumpeterian effect for the follower and the avoid competition effect for the leader. This would decrease their incentives to invest in R&D and decrease growth rates in other than neck-and-neck states. Hence, the overall outcome is not clear.

Also the robustness of the results with respect to different models of industry dynamics, i.e. different sources of firms' variety like the degree of substitution, extent of fixed costs etc. would be interesting to check. Another natural extension would be to allow for entry and exit. Exit would bound the industry's gap and encourages predatory behavior. This would be a kind of an endpoint effect and rise the incentives to move ahead for leaders. Allowing for (re-) entry by making it possible to copy the incumbent's technology at certain cost might promote efforts by the incumbent to gain so much experience that relative high R&D cost for a new firm deter entry. In such an extension the modeling of imitation and licensing would be crucial.

Another important area for future work is further detailed empirical investigation of the long run relation between incentives to innovate and market structure.

Appendix

A No Acquisition of Knowledge

A.1 Optimal R&D Effort

The maximized Bellman equations are:[22]

$$\rho V_i(\Delta_i) = \pi_i(\Delta_i) - z(u_i(\Delta_i)) + [V_i(\Delta_i+1) - V_i(\Delta_i)]\, h_i(u_i(\Delta_i)) + [V_i(\Delta_i-1) - V_i(s)]\, h_j(u_j(\Delta_i)). \tag{15}$$

Using the envelope theorem and $\lambda = 1$ the first order condition for firm i yields

$$V_i(\Delta_i+1) - V_i(\Delta_i) = u_i(\Delta_i)^{\eta-1}. \tag{16}$$

Note that each R&D effort is proportional to the incremental value that would result from innovating.[23] Inserting this in (15) gives

$$V_i(\Delta_i) = \tfrac{1}{\rho}\Big(\pi_i(\Delta_i) - z(u_i(\Delta_i)) - z'(u_i(\Delta_i))u_i(\Delta) - z'(u_i(\Delta_i-1))u_j(\Delta_i)\Big) \tag{17}$$

and

$$\begin{aligned} V_i(\Delta_i+1) = \tfrac{1}{\rho}\Big(&\pi_i(\Delta_i+1) - z(u_i(\Delta_i+1)) - z'(u_i(\Delta_i+1))u_i(\Delta_i+1) \\ &- z'(u_1(\Delta_i))u_j(\Delta_i+1)\Big). \end{aligned} \tag{18}$$

Using this in the first order condition yields

$$\begin{aligned} &u_i(\Delta_i)^{\eta-1}\,(\rho + u_i(\Delta_i) - u_j(\Delta_i+1)) \\ &\quad = \pi_i(\Delta_i+1) - z(u_i(\Delta_i+1)) - \pi_i(\Delta_i) + z(u_i(\Delta_i)) + u_i(\Delta_i+1)^\eta - (u_i(\Delta_i-1))^{\eta-1}u_j(\Delta_i). \end{aligned} \tag{19}$$

As the firms are ex ante symmetric, $u_i(\Delta_i) = u_j(-\Delta_i)$ holds. This yields the reduced form R&D equations

$$\begin{aligned} &u_i(\Delta_i)^{\eta-1}\,(\rho + u_i(\Delta_i) - u_i(-\Delta_i-1)) \\ &\quad = \pi_i(\Delta_i+1) - z(u_i(\Delta_i+1)) - \pi_i(\Delta_i) + z(u_i(\Delta_i)) + u_i(\Delta_i+1)^\eta - (u_i(\Delta_i-1))^{\eta-1}u_i(-\Delta_i). \end{aligned} \tag{20}$$

For the special case of $\Delta_i = 0$, this simplifies to

$$\begin{aligned} &u_i(0)^{\eta-1}\,(\rho + u_i(0) - u_1(-1)) \\ &\quad = 1 - e^{-1} - \frac{1}{\eta}\,(u_i(1)^\eta - u_i(0)^\eta) + u_i(1)^\eta - (u_i(-1))^{\eta-1}u_i(0). \end{aligned} \tag{21}$$

Solving this for $u(0)$[24] and assuming $\eta = 2$ yields

$$u(0) = \sqrt{2 - \frac{2}{e} + \rho^2 + u(1)^2} - \rho. \tag{22}$$

For $\Delta > 0$ we have $\pi(\Delta+1) - z(u(\Delta+1)) - \pi(\Delta) + z(u(\Delta)) = e^{-\Delta-1}(1-e) + \frac{1}{\eta}\,(u(\Delta)^\eta - u(\Delta+1)^\eta)$ while when $\Delta < 0$ the increase in profit when moving one step ahead is zero, i.e. $\pi(u(\Delta+1)) - \pi(u(\Delta)) = 0$. Using this we get the following relations of optimal R&D stated in lemma 1.

[22] The derivation of the maximized Bellman equations for the general case including knowledge accumulation is provided in section B of the appendix.

[23] For the special case $\eta = 2$ effort is even strictly proportional to the incremental value.

[24] For the sake of readability we will suppress the identity of the firm where not necessary.

A.2 Pattern of Equilibrium Effort

To simplify the analysis we assume $\rho = 0$.[25] Furthermore, we restrict attention to non-fluctuating pattern. That means for all $\Delta > 0$: if $u(\Delta) > u(\Delta+1)$ then $u(\Delta+1) > u(\Delta+2)$ and vice versa. The same has to be true for the follower, i.e. for all $-\Delta < 0$: if $u(-\Delta) > u(-\Delta-1)$ then $u(-\Delta-1) > u(-\Delta-2)$ and vice versa. Hence, we are not able to rigorously prove the uniqueness of the equilibrium we derive. However, numerical simulations show that equilibria with a "noisy" pattern do not exist and the equilibrium we analytically find is indeed unique. Besides, economic intuitions for fluctuating effort do not exist when time preference rate is set to zero.

From part 1 of Lemma 1, obviously $u(0) > u(1)$ results.

To see how the effort of the industry's follower reacts on technological jumps we firstly analyze the equation given in part 3 of lemma 1.

This equation gives for $-\Delta < 0$

$$\begin{aligned}\left(u(-\Delta) - \tfrac{1}{2}u(\Delta-1)\right)^2 = &\left(u(-\Delta+1) - \tfrac{1}{2}u(\Delta-1)\right)^2 \\ &+ u(-\Delta+1)u(\Delta-1) - u(-\Delta-1)u(\Delta)\end{aligned}$$

and hence

$$\begin{aligned}&\text{sign}\{(u(-\Delta) - \tfrac{1}{2}u(\Delta-1))^2 - (u(-\Delta+1) - \tfrac{1}{2}u(\Delta-1))^2\} \\ &\quad = \text{sign}\{u(-\Delta+1)u(\Delta-1) - u(-\Delta-1)u(\Delta)\}.\end{aligned}$$

Here, we have to distinct different cases. Let us first assume that $u(-\Delta) > \frac{1}{2}u(\Delta)\ \forall \Delta > 0$ (implying $u(-\Delta+1) > \frac{1}{2}u(\Delta-1)\ \forall\Delta > 0$). Then, we get

$$\begin{aligned}&\text{sign}\{u(-\Delta) - u(-\Delta+1)\} \\ &\quad = \text{sign}\{u(-\Delta+1)u(\Delta-1) - u(-\Delta-1)u(\Delta)\}. \end{aligned} \tag{23}$$

This means efforts for the follower are decreasing with the gap when $u(-\Delta+1)u(\Delta-1) < u(-\Delta-1)u(\Delta)$ holds. The opposite is true for $u(-\Delta) < \frac{1}{2}u(\Delta)\ \forall\Delta > 0$, i.e. follower's effort increases if $u(-\Delta+1)u(\Delta-1) > u(-\Delta-1)u(\Delta)$.

For $\Delta = 1$, these relations also compare the follower's effort with the effort of a neck-and-neck firm. Analyzing the situation for $-\Delta = -1$ yields

$$\text{sign}\{u(-1) - u(0)\} = \text{sign}\{u(0)^2 - u(-2)u(1)\}.$$

We already know that investment in neck-and-neck state is higher than effort of a firm being one step ahead, i.e. $u(0) > u(1)$. This indicates that $u(0)^2 > u(-2)u(1)$ might hold, implicating $u(-1) > u(0)$. This is true as long as $u(-2)$ is not too large, i.e. $u(-1) > u(0) > u(1)$ iff $u(-2) < \frac{u(0)^2}{u(1)} > u(0)$. If $u(-2)$ is large enough to outweigh the difference between $u(0)$ and $u(1)$ we have $u(-1) > u(0)$. In that case the relation $u(-2) > u(0) > u(-1)$ holds. That means the optimal patterns shows some kind of fluctuation.

We can illustrate the characteristics of the general equation (23) by means of the example of $\Delta = 2$. Equation (23) yields

$$\text{sign}\{u(-2) - u(-1)\} = \text{sign}\{u(-1)u(1) - u(-3)u(2)\}.$$

[25] We were able to show that all results hold with ρ in quality. However, the formal analysis is excessively more complex.

As we are not looking for fluctuating patterns, we either have $u(-3) > u(-2) > u(-1)$ or $u(-3) < u(-2) < u(-1)$. In the first case, $u(2)$ would have to be sufficiently small to ensure $u(-1)u(1) > u(-3)u(2)$. In the case of $u(-3) < u(-2) < u(-1)$, $u(2)$ needs to be sufficiently large. Obviously, in either case do R&D efforts for leader and follower go into opposite directions when the gap increases. This clearly also holds true for the general case of $\Delta > 2$ and equation (23). More precisely, $u(-\Delta) \gtreqless u(-\Delta+1)$ if $\frac{u(-\Delta+1)}{u(-\Delta-1)} \gtreqless \frac{u(\Delta)}{u(\Delta-1)}$. If now $u(\Delta) > u(\Delta-1)$, $u(-\Delta+1) > u(-\Delta) > u(-\Delta-1)$ has to hold and vice versa.

Keeping in mind that leader's and follower's effort move into opposite directions when the gap increases, let's now look on the leader's optimal effort given in part 2 of the lemma. We can directly see that for $\Delta > 0$

$$\begin{aligned}(u(\Delta) - u(-\Delta-1))^2 = &(u(\Delta+1) - u(-\Delta-1))^2 \\ &+ 2\left(u(-\Delta-1)u(\Delta+1) - u(\Delta-1)u(-\Delta) + e^{-\Delta}\left(1-\tfrac{1}{e}\right)\right),\end{aligned}$$

and hence

$$\begin{aligned}&\text{sign}\{(u(\Delta) - u(-\Delta-1))^2 - (u(\Delta+1) - u(-\Delta-1))^2\} \\ &\quad = \text{sign}\{u(-\Delta-1)u(\Delta+1) - u(\Delta-1)u(-\Delta) + e^{-\Delta}\left(1-\tfrac{1}{e}\right)\}\end{aligned}$$

holds. Again, we have to distinct different cases. Let us first assume $u(\Delta) < u(-\Delta-1)$ and $u(\Delta) < u(-\Delta)\ \forall\Delta > 0$. In that case the assumption $u(-\Delta) > \frac{1}{2}u(\Delta)\ \forall\Delta > 0$ holds as well. Besides, as we know leader's and follower's effort move into opposite directions, the assumptions implicate $u(\Delta') < u(-\Delta)\ \forall\Delta, \Delta' > 0$.

Then, we get

$$\begin{aligned}&\text{sign}\{u(\Delta+1) - u(\Delta)\} \\ &\quad = \text{sign}\{u(-\Delta-1)u(\Delta+1) - u(\Delta-1)u(-\Delta) + e^{-\Delta}\left(1-\tfrac{1}{e}\right)\}\end{aligned} \tag{24}$$

That means, we see increasing efforts for leaders when $u(-\Delta-1)u(\Delta+1) > u(-\Delta)u(\Delta-1) - e^{-\Delta}\left(1-\frac{1}{e}\right)$.

In the case of $u(\Delta') < u(-\Delta)\ \forall\Delta, \Delta' > 0$ the leader's effort can only be increasing if the follower's effort is decreasing and furthermore if $u(-\Delta-1)u(\Delta+1) > u(-\Delta)u(\Delta-1) - e^{-\Delta}\left(1-\frac{1}{e}\right)$ holds. As in this case $u(\Delta+1) > u(\Delta-1)$ holds, we have found an equilibrium where the laggard's increase in effort with an increase in gap is not too large. Besides, it is clear that beyond this $u(-1) > u(0) > u(1)$ holds, since effort is decreasing for the follower and therefore $u(-2) > u(0) > u(-1)$ cannot hold. The resulting pattern is that summarized in terms of investment by proposition 2. Hence, we have shown that the described optimal behavior is indeed an equilibrium.

To show that no other equilibria exist is a very comprehensive task and needs quantifying analysis. Unfortunately we are not able to analytically show the uniqueness of the equilibrium. However, numerical simulations show that the derived equilibrium is indeed unique.

B The Effect of Knowledge

To solve for the Markov-stationary equilibrium we use dynamic programming methods and therefore derive the Bellman equations. Defining the optimal programs for the firms $i = 1, 2$

as $V_i(s) \equiv \max_{\{z_i(\tau)\}} \Pi_i(s(t))$ s.t. the evolutions of the state variables $s \equiv (k_1, k_2, x_1, x_2)$, the Bellman equations are given by

$$\rho V_i(s(t)) = \max_{z_i(t)} \left\{ \pi_i(s(t)) - z_i(t)) + \frac{1}{dt} E_t dV_i(s(t)) \right\},$$

where the R&D effort of the competitor is taken as given. Given this general form we compute the differential $dV_i(s(t))$ given the evolutions of the state variables and form expectations. This yields

$$\begin{aligned} E_t dV_i(s(t)) &= \left[\frac{\partial V_i(s)}{\partial k_i} (u(z_i) - \delta k_i) + \frac{\partial V_i(s)}{\partial k_j} (u(z_j) - \delta k_j) \right] dt \\ &\quad + [V_i(x_i + 1, \cdot) - V_i(s)] h_i(z_i) dt + [V_i(x_j + 1, \cdot) - V_i(s)] h_j(z_j) dt. \end{aligned}$$

The Bellman equations therefore read

$$\begin{aligned} \rho V_i(s(t)) = \max_{z_i(t)} \Bigg\{ & \pi_i(s(t)) - z_i(t)) + \left[\frac{\partial V_i(s)}{\partial k_i} (u(z_i) - \delta k_i) + \frac{\partial V_i(s)}{\partial k_j} (u(z_j) - \delta k_j) \right] \\ & + [V_i(x_i + 1, \cdot) - V_i(s)] h_i(z_i) + [V_i(x_j + 1, \cdot) - V_i(s)] h_j(z_j) \Bigg\}. \end{aligned} \tag{25}$$

Then, the first-order conditions are

$$-1 + \frac{\partial V_i(s)}{\partial k_i} u'(z_i) + [V_i(x_i + 1, \cdot) - V_i(s)] h_i'(z_i) \overset{!}{=} 0 \tag{26}$$

$$\Leftrightarrow \frac{\partial V_i(s)}{\partial k_i} = \frac{1 - \lambda [V_i(x_i + 1, \cdot) - V_i(s)]}{u'(z_i)}. \tag{27}$$

Current gain from not investing an additional unit, i.e. -1 , must equal future gain from an additional unit of investment which is influenced by the change of knowledge stock (through the increase in effort) and the probability of a successful innovation.

This yields

$$\begin{aligned} d\frac{\partial V_i(s)}{\partial k_i} &= -d\frac{1 - \lambda [V_i(x_i + 1, \cdot) - V_i(s)]}{u'(z_i)} \\ &= d\frac{1}{u'(z_i)} - \lambda d \frac{V_i(x_i + 1, \cdot) - V_i(s)}{u'(z_i)}. \end{aligned}$$

In the next step, we state the maximized Bellman equations from (25) as the Bellman equations where controls are replaced by their optimal values:

$$\begin{aligned} \rho V_i(s) &= \pi_i(z_i(s)) + \frac{\partial V_i(s)}{\partial k_i} (u(z_i(s)) - \delta k_i) + \frac{\partial V_i(s)}{\partial k_j} (u(z_j(s)) - \delta k_j) \\ &\quad + [V_i(x_i + 1, \cdot) - V_i(s)] h_i(z_i(s)) + [V_i(x_j + 1, \cdot) - V_i(s)] h_j(z_j(s)). \end{aligned} \tag{28}$$

We now compute the derivatives with respect to the state variables k_i using the envelope theorem on the Bellman equation (25). This gives expressions for the shadow prices $\frac{\partial V_i(s)}{\partial k_i}$,

$$\begin{aligned} \rho \frac{\partial V_i(s)}{\partial k_i} &= \frac{\partial^2 V_i(s)}{\partial k_i^2} (u(z_i(s)) - \delta k_i) - \delta \frac{\partial V_i(s)}{\partial k_i} + \frac{\partial^2 V_i(s)}{\partial k_i \partial k_j} (u(z_j(s)) - \delta k_j) \\ &\quad + \left[\frac{\partial V_i(x_i + 1, \cdot)}{\partial k_i} - \frac{\partial V_i(s)}{\partial k_i} \right] h_i(z_i(s)) + [V_i(x_i + 1, \cdot) - V_i(s)] \alpha \gamma k_i^{\alpha - 1} \\ &\quad + \left[\frac{\partial V_i(x_j + 1, \cdot)}{\partial k_i} - \frac{\partial V_i(s)}{\partial k_i} \right] h_j(z_j(s)), \end{aligned} \tag{29}$$

Furthermore,

$$
\begin{aligned}
\rho\frac{\partial V_i(s)}{\partial k_j} &= \frac{\partial^2 V_i(s)}{\partial k_i \partial k_j}\left(u(z_i(s)) - \delta k_i\right) + \frac{\partial^2 V_i(s)}{\partial k_j^2}\left(u(z_j(s)) - \delta k_j\right) - \delta\frac{\partial V_i(s)}{\partial k_j} \\
&+ \left[\frac{\partial V_i(x_i+1,\cdot)}{\partial k_j} - \frac{\partial V_i(s)}{\partial k_j}\right] h_i(z_i(s)) \\
&+ \left[\frac{\partial V_i(x_j+1,\cdot)}{\partial k_j} - \frac{\partial V_i(s)}{\partial k_j}\right] h_j(z_j(s)) + \left[V_i(x_j+1,\cdot) - V_i(s)\right]\alpha\gamma k_j^{\alpha-1}.
\end{aligned}
$$

Given the evolutions of the state variables we can compute the differentials of the shadow prices:

$$
\begin{aligned}
d\frac{\partial V_i(s)}{\partial k_i} &= \left[\frac{\partial^2 V_i(s)}{\partial k_i^2}\left(u(z_i) - \delta k_i\right) + \frac{\partial^2 V_i(s)}{\partial k_i \partial k_j}\left(u(z_j) - \delta k_j\right)\right] dt \\
&+ \left[\frac{\partial V_i(x_i+1,\cdot)}{\partial k_i} - \frac{\partial V_i(s)}{\partial k_i}\right] dq_i(s) + \left[\frac{\partial V_i(x_j+1,\cdot)}{\partial k_i} - \frac{\partial V_i(s)}{\partial k_i}\right] dq_j(s), \quad (30)
\end{aligned}
$$

and

$$
\begin{aligned}
d\frac{\partial V_i(s)}{\partial k_j} &= \left[\frac{\partial^2 V_i(s)}{\partial k_i \partial k_j}\left(u(z_i) - \delta k_i\right) + \frac{\partial^2 V_i(s)}{\partial k_j^2}\left(u(z_j) - \delta k_j\right)\right] dt \\
&+ \left[\frac{\partial V_i(x_i+1,\cdot)}{\partial k_j} - \frac{\partial V_i(s)}{\partial k_j}\right] dq_i(s) + \left[\frac{\partial V_i(x_j+1,\cdot)}{\partial k_j} - \frac{\partial V_i(s)}{\partial k_j}\right] dq_j(s), \quad (31)
\end{aligned}
$$

Replacing $\frac{\partial^2 V_i(s)}{\partial k_i^2}\left(u(z_i) - \delta k_i\right) + \frac{\partial^2 V_i(s)}{\partial k_i \partial k_i}\left(u(z_j) - \delta k_j\right)$ in (30) by the same expressions from (29) gives

$$
\begin{aligned}
d\frac{\partial V_i(s)}{\partial k_i} =& \Bigg\{(\rho+\delta)\frac{\partial V_i(s)}{\partial k_i} - \left[\frac{\partial V_1(x_i+1,\cdot)}{\partial k_i} - \frac{\partial V_i(s)}{\partial k_i}\right] h_i(z_i(s)) \\
&- \left[\frac{\partial V_i(x_j+1,\cdot)}{\partial k_i} - \frac{\partial V_i(s)}{\partial k_1}\right] h_j(z_j(s)) - \left[V_i(x_i+1,\cdot) - V_i(s)\right]\alpha\gamma k_i^{\alpha-1}\Bigg\} dt \\
&+ \left[\frac{\partial V_i(x_i+1,\cdot)}{\partial k_i} - \frac{\partial V_i(s)}{\partial k_i}\right] dq_i(s) + \left[\frac{\partial V_i(x_j+1,\cdot)}{\partial k_i} - \frac{\partial V_i(s)}{\partial k_i}\right] dq_j(s), \qquad (32)
\end{aligned}
$$

and

$$
\begin{aligned}
d\frac{\partial V_i(s)}{\partial k_j} =& \Bigg\{(\rho+\delta)\frac{\partial V_i(s)}{\partial k_j} + \left[\frac{\partial V_i(x_i+1,\cdot)}{\partial k_j} - \frac{\partial V_i(s)}{\partial k_j}\right] h_i(z_i(s)) \\
&+ \left[\frac{\partial V_i(x_j+1,\cdot)}{\partial k_j} - \frac{\partial V_i(s)}{\partial k_j}\right] h_j(z_j(s)) + \left[V_i(x_j+1,\cdot) - V_i(s)\right]\alpha\gamma k_j^{\alpha-1}\Bigg\} dt \\
&+ \left[\frac{\partial V_i(x_i+1,\cdot)}{\partial k_j} - \frac{\partial V_i(s)}{\partial k_j}\right] dq_i(s) + \left[\frac{\partial V_i(x_j+1,\cdot)}{\partial k_j} - \frac{\partial V_i(s)}{\partial k_j}\right] dq_j(s), \qquad (33)
\end{aligned}
$$

Finally, we replace the marginal values by marginal profits from the first order conditions

(26).

$$
\begin{aligned}
& d\frac{1}{u'(z_i)} - \lambda d\frac{V_i(x_i+1,\cdot)-V_i(s)}{u'(z_i(s))} \\
=& \Bigg\{ (\rho+\delta)\frac{1-\lambda\left[V_i(x_i+1,\cdot)-V_i(s)\right]}{u'(z_i)} \\
& - \left[\frac{1-\lambda\left[V_i(x_i+2,\cdot)-V_i(x_i+1)\right]}{u'(z_i(x_i+1))} - \frac{1-\lambda\left[V_i(x_i+1,\cdot)-V_i(s)\right]}{u'(z_i)}\right] h_i(z_i(s)) \\
& - \left[\frac{1-\lambda\left[V_i(\cdot)-V_i(x_{-i}+1)\right]}{u'(z_i(x_{-i}+1))} - \frac{1-\lambda\left[V_i(x_i+1,\cdot)-V_i(s)\right]}{u'(z_i)}\right] h_j(z_j(s)) \\
& - \left[V_i(x_i+1,\cdot)-V_i(s)\right]\alpha\gamma k_i^{\alpha-1}\Bigg\} dt \\
& + \left[\frac{1-\lambda\left[V_i(x_i+2,\cdot)-V_i(x_i+1)\right]}{u'(z_i(x_i+1))} - \frac{1-\lambda\left[V_i(x_i+1,\cdot)-V_i(s)\right]}{u'(z_i)}\right] dq_i(s) \\
& + \left[\frac{1-\lambda\left[V_i(\cdot)-V_i(x_{-i}+1)\right]}{u'(z_i(x_{-i}+1))} - \frac{1-\lambda\left[V_i(x_i+1,\cdot)-V_i(s)\right]}{u'(z_i)}\right] dq_j(s). \qquad (34)
\end{aligned}
$$

Due to the modeling approach only the technological gap and not the technological levels as such matters for firms' values, i.e. the effect of the competitor moving one step forward is the same as moving one step backwards.

$$
\begin{aligned}
& d\frac{1}{u'(z_i)} - \lambda d\frac{V_i(x_i+1,\cdot)-V_i(s)}{u'(z_i(s))} \\
=& \Bigg\{ (\rho+\delta)\frac{1-\lambda\left[V_i(x_i+1,\cdot)-V_i(s)\right]}{u'(z_i)} \\
& - \left[\frac{1-\lambda\left[V_i(x_i+2,\cdot)-V_i(x_i+1)\right]}{u'(z_i(x_i+1))} - \frac{1-\lambda\left[V_i(x_i+1,\cdot)-V_i(s)\right]}{u'(z_i)}\right] h_i(z_i(s)) \\
& - \left[\frac{1-\lambda\left[V_i(\cdot)-V_i(x_{-i}+1)\right]}{u'(z_i(x_{-i}+1))} - \frac{1-\lambda\left[V_i(x_i+1,\cdot)-V_i(s)\right]}{u'(z_i)}\right] h_j(z_j(s)) \\
& - \left[V_i(x_i+1,\cdot)-V_i(s)\right]\alpha\gamma k_i^{\alpha-1}\Bigg\} dt \\
& + \left[\frac{1-\lambda\left[V_i(x_i+2,\cdot)-V_i(x_i+1)\right]}{u'(z_i(x_i+1))} - \frac{1-\lambda\left[V_i(x_i+1,\cdot)-V_i(s)\right]}{u'(z_i)}\right] dq_i(s) \\
& + \left[\frac{1-\lambda\left[V_i(\cdot)-V_i(x_{-i}+1)\right]}{u'(z_i(x_{-i}+1))} - \frac{1-\lambda\left[V_i(x_i+1,\cdot)-V_i(s)\right]}{u'(z_i)}\right] dq_j(s). \qquad (35)
\end{aligned}
$$

On the other hand we can use the maximized Bellman equations together with the first order condition and get

$$
\begin{aligned}
\rho V_i(s) \;=\; & \pi_i(s) - z_i(s) + \frac{1-\lambda\left[V_i(x_i+1,\cdot)-V_i(s)\right]}{u'(z_i)}\left(u(z_i(s))-\delta k_i\right) + \frac{\partial V_i(s)}{\partial k_j}\left(u(z_j(s))-\delta k_j\right) \\
& + \left[V_i(x_i+1,\cdot)-V_i(s)\right] h_i(z_i(s)) + \left[V_i(x_j+1,\cdot)-V_i(s)\right] h_j(z_j(s)). \qquad (36)
\end{aligned}
$$

In this step we make another simplifying assumption namely that investment in R&D does not influence a firm's probability of success immediately. Hence, with $\lambda = 0$ the first order conditions read

$$-1 + \frac{\partial V_i(s)}{\partial k_i} u'(z_i(s)) = 0, \qquad (37)$$

yielding

$$d\frac{\partial V_i(s)}{\partial k_i} = d\frac{1}{u'(z_i(s))}. \qquad (38)$$

Therefore, we get the optimal rule describing the evolution of marginal profits:

$$
\begin{aligned}
-d\frac{-1}{u'(z_i(s))} =& \Bigg\{-(\rho+\delta)\frac{-1}{u'(z_i(s))} + \left[\frac{-1}{u'(z_i(x_i+1,\cdot))} - \frac{-1}{u'(z_i(s))}\right]\gamma k_i^{\alpha} \\
&+ \left[\frac{-1)}{u'(z_i(x_j+1,\cdot))} - \frac{-1}{u'(z_i(s))}\right]\gamma k_j^{\alpha} - \left[V_i(x_i+1,\cdot) - V_i(s)\right]\alpha\gamma k_i^{\alpha-1}\Bigg\} dt \\
&+ \left[\frac{-1}{u'(z_i(s))} - \frac{-1}{u'(z_i(x_i+1,\cdot))}\right] dq_i(s) \\
&+ \left[\frac{-1}{u'(z_i(s))} - \frac{-1}{u'(z_i(x_j+1,\cdot))}\right] dq_j(s), \qquad (39)
\end{aligned}
$$

The rule shows how marginal profit changes in a deterministic and stochastic way. While there is a one-to-one mapping from marginal profit to investment which allows some inferences about investment from (39), it would be more useful to have a rule for optimal investment itself. With firms' instantaneous profits (3) and rate of knowledge acquisition $u(z_i) = (\eta z_i)^{\frac{1}{\eta}}$ we get

$$
d\frac{-1}{u'(z_i(s))} = -\frac{\partial \frac{1}{u'(z_i(s))}}{\partial z_i(s)} dz_i(s) = \frac{u''(z_i(s))}{u'(z_i(s))^2} dz_i(s) = (1-\eta)(\eta z_i(s))^{-\frac{1}{\eta}} dz_i(s)
$$

and hence

$$
\begin{aligned}
dz_i(s) \quad = \quad & \frac{(\eta z_i(s))^{\frac{1}{\eta}}}{\eta - 1}\Bigg[\Bigg\{\frac{(\rho+\delta)}{u'(z_i(s))} + \left[\frac{1}{u'(z_i(s))} - \frac{1}{u'(z_i(x_i+1,\cdot))}\right]\gamma k_i^{\alpha} \\
&+ \left[\frac{1}{u'(z_i(s))} - \frac{1}{u'(z_i(x_i-1,\cdot))}\right]\gamma k_j^{\alpha} - \left[V_i(x_i+1,\cdot) - V_i(s)\right]\alpha\gamma k_i^{\alpha-1}\Bigg\} dt \\
&+ \left[\frac{1}{u'(z_i(x_i+1,\cdot))} - \frac{1}{u'(z_i(s))}\right] dq_i(s) \\
&+ \left[\frac{1}{u'(z_i(x_i-1,\cdot))} - \frac{1}{u'(z_i(s))}\right] dq_j(s)\Bigg]. \qquad (40)
\end{aligned}
$$

These rules describe the evolution of investment under optimal behavior for the firms. Growth of investment depends on the right-hand side in a deterministic way on the typical sum of the depreciation and time preference rate per marginal rate of knowledge acquisition plus the "k-terms" which capture the impact of uncertainty. To understand the meaning of these terms we analyze whether investment jumps up or down, following a jump of the own or competitor's technology. Since $\eta > 1$ the term $\frac{1}{u'(z_1(s))} - \frac{1}{u'(z_1(x_1+1,\cdot))} = \eta^{\frac{\eta-1}{\eta}}\left(z_1(s)^{\frac{\eta-1}{\eta}} - z_1(x_1+1,\cdot)^{\frac{\eta-1}{\eta}}\right)$ is negative if $z_1(s) < z_1(x_1+1,\cdot)$. If this is the case, investment increases slower (or decreases even faster) if the probability of a jump of the own technology due to a higher knowledge stock is high. On the other hand investment increases faster (or decreases slower) if the probability of a jump of the competitor's technology due to his higher knowledge stock is high.

The dq_{xi}-terms give discrete changes in the case of a jump in x_i. When x_i jumps and $dq_{x_i(s)} = 1$ ($dq_{x_j(s)} = 0$, i.e. there is no contemporaneous jump in x_j) and $dt = 0$ for this small instant of the jump, equation (40) says that $dz_i(s)$ on the left hand side is given by

$$
\frac{\eta z_i(s)}{\eta - 1}\left(z_i(s)^{\frac{1-\eta}{\eta}} z_i(x_i+1,\cdot)^{\frac{\eta-1}{\eta}} - 1\right) \qquad (41)
$$

on the right hand side. This is positive as long as $z_i(s) < z_i(x_i+1,\cdot)$ which is consistent with the definition of $dz_i(s)$ given by $z_i(x_i+1,\cdot) - z_i(s)$. Solving this for $z_i(s)$ interestingly yields $z_i(s) = z_i(x_i+1,\cdot)$. Hence, optimal investment does not immediately react to a jump in the industry's state. This result is stated in proposition 4.

Using the derived fact, from (40) we can determine the evolution of optimal investment:

$$\frac{dz_i(s)}{dt} = z_i(s)\frac{\eta}{\eta-1}\Big(\rho+\delta-(\eta z_i(s))^{\frac{1-\eta}{\eta}}\underbrace{[V_i(x_i+1,\cdot)-V_i(s)]}_{\equiv\Phi_i(s)}\alpha\gamma k_i^{\alpha-1}\Big). \tag{42}$$

We can not determine the value of $\Phi_i(s)$, but we know it is always positive and from the first order condition we get $\frac{\partial\Phi_i(s)}{\partial k_i}=0$. Thus, $\Phi(\cdot)$ is only a function of the technological gap Δ and the same function for both firms, i.e $\Phi_i(\Delta)=\Phi_j(-\Delta)$. Hence, we can write $\Phi(\Delta_i)\equiv\Phi_i(\Delta)$. With this result it can directly be seen that more knowledge a firm has acquired the smaller is the growth rate of optimal investment.

Furthermore, from (36) and with $\lambda=0$ we can analyze the shape of $\Phi(\Delta)$. As the value function inherits its shape from the profit function, the value function will be bounded from below and above and will converge to these bound for $\Delta\to-\infty$ and $\Delta\to\infty$ respectively. Hence, $\frac{\partial(V_i(\Delta+1)-V_i(\Delta))}{\partial\Delta}$ is negative for high values of Δ and negative for small values.

As the slope of the value function measures a leader's incentive to innovate, this slope is maximal around the neck and neck point since neck-and-neck firms perform R&D at a higher intensity than industry leaders.

The maximized Bellman equations (36) hold for all optimal efforts, especially at steady state as well, i.e. when $u(z_j)-\delta k_j=0$. In this case, for $\Delta<0$ we have:

$$\rho V_i(s) = -z_i(s)+[\Phi(\Delta)]\,\gamma k_i^\alpha-[\Phi(\Delta-1)]\,\gamma k_j^\alpha.$$

The right hand side can only be positive for positive $\Phi(\Delta)$ if $\Phi(\Delta)>\Phi(\Delta-1)$ and therefore we have $\frac{\partial\Phi(\Delta)}{\partial\Delta}>0$ for $\Delta<0$.

The derivative of the maximized Bellman equation (36) with respect to Δ for $\Delta<0$ using the envelope theorem gives

$$\begin{aligned}\rho\frac{\partial V_i(\Delta,k_i,k_j)}{\partial\Delta} &= \frac{\partial^2 V_i(\Delta,\cdot)}{\partial\Delta\partial k_j}(u(z_j(s))-\delta k_j)+\left[\frac{\partial V_i(\Delta+1,\cdot)}{\partial\Delta}-\frac{\partial V_i(\Delta,\cdot)}{\partial\Delta}\right]\gamma k_i^\alpha \\ &\quad+\left[\frac{\partial V_i(\Delta-1,\cdot)}{\partial\Delta}-\frac{\partial V_i(\Delta,\cdot)}{\partial\Delta}\right]\gamma k_j^\alpha. \end{aligned}\tag{43}$$

For $\Delta>0$ the derivative yields

$$\begin{aligned}\rho\frac{\partial V_i(\Delta,k_i,k_j)}{\partial\Delta} &= e^{-\Delta}+\frac{\partial^2 V_i(\Delta,\cdot)}{\partial\Delta\partial k_j}(u(z_j(s))-\delta k_j)+\left[\frac{\partial V_i(\Delta+1,\cdot)}{\partial\Delta}-\frac{\partial V_i(\Delta,\cdot)}{\partial\Delta}\right]\gamma k_i^\alpha \\ &\quad+\left[\frac{\partial V_i(\Delta-1,\cdot)}{\partial\Delta}-\frac{\partial V_i(\Delta,\cdot)}{\partial\Delta}\right]\gamma k_j^\alpha. \end{aligned}\tag{44}$$

Here, we can see that the only critical value of Δ, i.e. a value where signs could possibly change, is indeed at $\Delta=0$.

We can now determine the derivative of the value effect of a technological step ahead for $\Delta>0$:

$$\begin{aligned}&\rho\frac{\partial\left(V_i(\Delta+1,k_i,k_j)-V_i(\Delta,k_i,k_j)\right)}{\partial\Delta}\\ &=e^{-\Delta-1}(1-e)+\left(\frac{\partial^2 V_i(\Delta+1,\cdot)}{\partial\Delta\partial k_{-i}}-\frac{\partial^2 V_i(\Delta,\cdot)}{\partial\Delta\partial k_{-i}}\right)(u(z_j)-\delta k_j)\\ &+\left[\frac{\partial V_i(\Delta+2,\cdot)}{\partial\Delta}-2\frac{\partial V_i(\Delta+1,\cdot)}{\partial\Delta}+\frac{\partial V_i(\Delta,\cdot)}{\partial\Delta}\right]\gamma k_i^\alpha\\ &+\left[\frac{\partial V_i(\Delta,\cdot)}{\partial\Delta}-\frac{\partial V_i(\Delta+1,\cdot)}{\partial\Delta}-\frac{\partial V_i(\Delta-1,\cdot)}{\partial\Delta}+\frac{\partial V_i(\Delta,\cdot)}{\partial\Delta}\right]\gamma k_j^\alpha.\end{aligned}\tag{45}$$

We know $V(\cdot)$ is approaching the upper bound for $\Delta \to \infty$. Hence, for sufficiently large values of Δ, the second derivative of the value function and thus the fist derivative of $\Phi(\Delta)$ will be negative. But even more, we see that the right hand side of equation (45) is negative for all $\Delta > 0$ in steady state with sufficiently low knowledge stocks as $e^{-\Delta-1}(1-e) < 0$. Hence, the value function's inflection point has to be at $\Delta = 0$ and we have $\frac{\partial \Phi(\Delta)}{\partial \Delta} < 0$ for $\Delta > 0$.

References

Acemoglu, D., & Akcigit, U. (2006). *State-dependent intellectual property rights policy* (Working Paper No. 12775). National Bureau of Economic Research.

Aghion, P., Bloom, N., Blundell, R., Griffith, R., & Howitt, P. (2005). Competition and innovation: An inverted-u relationship. *Quarterly Journal of Economics*, *120*(2), p701 - 728.

Aghion, P., Harris, C., Howitt, P., & Vickers, J. (2001). Competition, imitation and growth with step-by-step innovation. *Review of Economic Studies*, *68*.

Argote, L., Beckman, S. L., & Epple, D. (1990). The persistence and transfer of learning in industrial settings. *Manage. Sci.*, *36*(2), 140–154.

Belleflamme, P., & Vergari, C. (2006). *Incentives to innovate in oligopolies* (Université catholique de Louvain, Département des Sciences Economiques Working Paper No. 2006008). Université catholique de Louvain, Département des Sciences Economiques.

Benkard, C. L. (2000). Learning and forgetting: The dynamics of aircraft production. *The American Economic Review*, *90*(4), 1034–1054.

Bental, B., & Peled, D. (1996). The accumulation of wealth and the cyclical generation of new technologies: A search theoretic approach. *International Economic Review*, *37*(3), 687–718.

Budd, C., Harris, C., & Vickers, J. (1993). A model of the evolution of duopoly: Does the asymmetry between firms tend to increase or decrease? *The Review of Economic Studies*, *60*(3), 543–573.

Cabral, L. M. B., & Riordan, M. H. (1994). The learning curve, market dominance, and predatory pricing. *Econometrica*, *62*(5), 1115–1140.

Cabral, L. M. B., & Riordan, M. H. (1997). The learning curve, predation, antitrust, and welfare. *The Journal of Industrial Economics*, *45*(2), 155–169.

Cohen, W. M., & Levin, R. C. (1989). Empirical studies of innovation and market structure. In R. Schmalensee & R. Willig (Eds.), *Handbook of industrial organization.*

Czarnitzki, D., & Kraft, K. (2004). An empirical test of the asymmetric models on innovative activity: who invests more into r&d, the incumbent or the challenger? *Journal of Economic Behavior & Organization*, *54*(2), 153-173.

Dasgupta, P., & Stiglitz, J. E. (1988). Learning-by-doing, market structure and industrial and trade policies. *Oxford Economic Papers*, *40*(2), 246-68.

Doraszelski, U. (2003). An r&d race with knowledge accumulation. *RAND Journal of Economics*, *34*(1), 20 - 42.

Ericson, R., & Pakes, A. (1995). Markov-perfect industry dynamics: A framework for empirical work. *Review of Economic Studies*, *62*(210), 53 - 82.

Francois, P., & Lloyd-Ellis, H. (2003). Animal spirits through creative destruction. *The American Economic Review*, *93*(3), 530–550.

Fudenberg, D., Gilbert, R., Stiglitz, J., & Tirole, J. (1983). Preemption, leapfrogging and competition in patent races. *European Economic Review*, *22*(1), 3-31.

Gabaix, X. (2005). *The granular origins of aggregate fluctuations* (2005 Meeting Papers No. 470). Society for Economic Dynamics.

Gilbert, R. (2006). Looking for mr. schumpeter: Where are we in the competition–innovation debate? *NBER Innovation Policy & the Economy*, *6*(1), p159 - 215.

Grossman, G. M., & Shapiro, C. (1987). Dynamic r&d competition. *Economic Journal*, *97*, p372 - 387.

Harris, C., & Vickers, J. (1985). Perfect equilibrium in a model of a race. *Review of Economic Studies*, *52*, p193 - 209.

Harris, C., & Vickers, J. (1987). Racing with uncertainty. *Review of Economic Studies*, *54*, p1 - 22.

Haruyama, T. (2005). *R&d policy in a volatile economy* (Working Paper). Kobe University.

Hörner, J. (2004). A perpetual race to stay ahead. *Review of Economic Studies*, *71*, p1064 - 1088.

Khanna, T. (1995). Racing behavior technological evolution in the high-end computer industry. *Research Policy*, *24*(6), 933-958.

Lerner, J. (1997). An empirical exploration of a technology race. *RAND Journal of Economics*, *28*(2), p228 - 247.

Lippman, S. A., & McCardle, K. F. (1988). Preemption in r&d races. *European Economic Review*, *32*(8), 1661-1669.

Maliar, L., & Maliar, S. (2004). Endogenous growth and endogenous business cycles. *Macroeconomic Dynamics*, *8*(05), 559-581.

Matsuyama, K. (1999). Growing through cycles. *Econometrica*, *67*(2), 335–347.

Mookherjee, D., & Ray, D. (1991). On the competitive pressure created by the diffusion of innovations. *Journal of Economic Theory*, *54*(1), 124-147.

Reinganum, J. F. (1989). The timing of innovation: Research, development, and diffusion. In R. Schmalensee & R. Willig (Eds.), *Handbook of industrial organization* (Vol. 1, p. 849-908). Elsevier.

Ross, S. M. (2003). *Introduction to probability models* (8. international ed.). Academic Press.

Segal, I., & Whinston, M. (2005). *Antitrust in innovative industries* (Working Paper No. 11525). National Bureau of Economic Research.

Sutton, J. (1991). *Sunk costs and market structure.* Cambridge, Mass.: MIT Press.

Sutton, J. (1998). *Technology and market structure: Theory and history.* Cambridge, Mass.: MIT Press.

Sutton, J. (2007). Market structure: Theory and evidence. In M. Armstrong & R. Porter (Eds.), *Handbook of industrial organization* (Vol. 3, p. 2301-2368). Elsevier.

Vives, X. (2006). *Innovation and competitive pressure* (IESE Research Papers No. D/634). IESE Business School.