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Microfinance and markets:
New results for the Besley-Coate group lending model

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Abstract

Microfinance currently experiences a huge inflow of private investors and a surge in the use of market instruments. This raises the question of what market equilibria in microfinance markets look like and which kinds of market failure tend to afflict them. The present paper conducts an equilibrium analysis of Besley and Coate's (1995) group lending model with enforcement problems. We show that a consideration of repayment rates alone is not sufficient to predict market outcomes, as it is biased towards group lending. Market equilibria are likely to exhibit the same kinds of market failure as equilibria in adverse selection models, viz., financial fragility, redlining, and credit rationing. Social sanctions ameliorate these problems, but do not eliminate them.

JEL classification: G21

Key words: microfinance, group lending, enforcement

1 Introduction

In one of the seminal models in the theory of microfinance, Besley and Coate (1995) (henceforth: "BC") investigate the impact of joint liability in borrower groups on loan repayment rates.[1] Returns are sufficient to repay with certainty in their model, but due to enforcement problems, borrowers do not repay unless the penalty for default weighs heavier than the burden of repayment. Group lending, as opposed to individual lending, has two effects on repayment rates then. For one thing, it enhances repayment when one borrower is able and willing to stand in for a member of a group who does not repay. For another, however, liability for the repayment of the other members of her group potentially discourages a borrower from repaying at all when she would have repaid an individual loan. BC's first key result is that, when payoffs are independently and uniformly distributed and penalties for default are proportional to payoffs, group lending leads to a higher repayment rate when the loan rate is sufficiently low, while the repayment rate is higher with individual lending when the loan rate is high. Their second main result is that social sanctions imposed on group members who refrain from making their contribution to the contractual repayment reduce the drawback of group lending. If the sanctions are severe enough, group lending yields the higher repayment rate.

BC emphasize that their results "should not be taken as implying that group lending is better or worse than individual lending in any broader sense than repayment rates" (p. 16), so "a more comprehensive analysis of the differences between the two lending schemes is an interesting subject for further research" (p. 16). This "requires a richer framework than provided by the model of this paper" (p. 15). The aim of the present paper is to supplement the BC model with a minimum set of additional assumptions that enables us to study the nature of equilibrium and the welfare effects of the different lending schemes.

The analysis is motivated by the recent huge inflow of private investors and the surge in the use of market instruments in the market for funding microfinance institutions (MFIs). According to Reille and Forster (2008, p. 1), "[t]he entry of private investors is the most notable change in the microfinance investment marketplace . . . Individuals and institutional investors – including international retail banks, investment banks, pension funds, and private equity funds – are all looking for ways to channel capital into microfinance, and investment banking techniques are being introduced to create investment vehicle alternatives that appeal to an increasingly broad range of investors." The total volume of the microfinance market is estimated at $25 billion

[1] See also Ghatak and Guinnane (1999, p. 209) and Armendariz de Aghion and Jonathan Morduch (2005, pp. 297-298).

in 2006. The top 150 MFIs are by now mature, mostly regulated, and profitable institutions, and a further 800 are set to follow.[2] These MFIs increasingly attract funds from individual and institutional investors, as opposed to development finance institutions (DFIs).[3] In particular, private investors make up for more than 50 percent of the $4 billion foreign investment in MFIs. A similar percentage of the cross-border investments is made not directly but via specialized microfinance investment vehicles (MIVs). Except blended-value funds (BVFs), most MIVs meet return expectations of about 5 percent in dollar terms.[4] Investment banks have started to securitize MFI claims in the form of CDOs. Dieckmann (2007, p. 10) suggests a back-of-the-envelope calculation that highlights the vast growth potential of the microcredit market: "While MFIs currently serve an estimated 100 million micro-borrowers, the total potential demand is roughly estimated at 1 bn" (given low penetration rates of below 3 percent in large markets such as India and Brazil). So this $25 billion market may grow ten-fold if it attracts the required funds. Exhausting this growth potential obviously necessitates a continuation of private capital inflows attracted by decent returns. So there is little doubt that the recent trend towards private investments and market instruments will continue over the foreseeable future.

The equilibrium analysis of the BC model brings forth several interesting results. To begin with, we consider the model without social sanctions. The equilibrium loan contract maximizes expected borrower utility subject to the constraint that the MFIs break even. We demonstrate that an equilibrium exists. In equilibrium, the borrowers get the finance needed for their projects or not, depending on the level of expected returns and the nature and severity of the penalties. Our first main result is that in an equilibrium with individual lending, it may be possible for MFIs to offer a group lending contract that has a lower interest rate, increases the repayment rate, and breaks even – but no borrower accepts this low-interest offer. This reflects the negative effect of group lending on repayment incentives highlighted by BC. Theoretically, this shows that repayment rates are an imperfect indicator for the viability of lending types in equilibrium, systematically biased towards group lending. Practically, this means that inflows of private capital will probably tend to crowd out group lending in favor of individual contracts.

Furthermore, we find that the return function (that relates the return on lending to the interest rate) is a hump-shaped function over the relevant range of interest rates. Accordingly, the equilibrium may be characterized by the same sorts of market failure that arise in adverse selection models with a hump-shaped return function (see Stiglitz and Weiss, 1981, Section 1, pp. 395-

[2] See Dieckmann (2007, pp. 6-7).

[3] Our definitions of MFIs and DFIs follows Reille and Forster (2008).

[4] See Reille and Forster (2008, Figure 1, p. 2, and Table 1, p. 7).

399). The equilibrium may be characterized by financial fragility, in that a small increase in the rate of return required by the investors (viz., a rise beyond the maximum of the return function) leads to a complete breakdown of the market (cf. Mankiw, 1986). From a cross-sectional perspective, when there are several microcredit markets of the BC type, redlining may occur: all borrowers get loans in some markets (those where the maximum of the return function is no less than the rate of return required by the investors), while no-one gets credit in other markets (cf. Riley, 1987). This result may be helpful in understanding why microfinance works well in some places but not in others. It implies that in order to maximize the total volume of credit given, DFIs should target the least profitable markets consistent with their return expectations, leaving the more profitable segments to private investors. Credit rationing may also arise: in a given market, some borrowers get funds, while other, indistinguishable, borrowers do not (cf. Stiglitz and Weiss, 1981). In sum, our second main result is that, irrespective of whether group lending or individual lending arises as the equilibrium mode of finance, microcredit markets with problems of enforcing repayments are likely to be characterized by the usual types of allocation problems encountered in loan markets with asymmetric information.[5]

We also analyze the model with social sanctions. Here our main result is that even if social sanctions are severe enough so that they eliminate the negative effect of group lending on repayment rates and group lending unambiguously becomes the equilibrium mode of finance, the allocation problems identified in the model without sanctions do not go away. That is, social sanctions ameliorate, but do not eliminate, the negative impact of enforcement problems on equilibria in microcredit markets.

The remainder of the paper is organized as follows. Section 2 describes the model without social sanctions. Section 3 summarizes BC's results on repayment rates. In Section 4, we go on to characterize the model equilibrium (details of the derivations are delegated to a technical appendix). Social sanctions are introduced in Section 5. Section 6 concludes.

2 Model

This section describes the model. We focus on the model with independently and uniformly distributed payoffs and proportional penalties for default. Since we do not alter any of BC's

[5] Arnold and Riley (2009) show that the return function cannot be hump-shaped in the original Stiglitz-Weiss (1981) model, so the types of equilibria sketched above cannot arise. The equilibrium outcome of the Stiglitz-Weiss (1981) model is an equilibrium with two interest rates, rationing at the lower rate, and market clearing at the higher rate.

assumptions, the exposition is kept brief. The additional assumptions made in our equilibrium analysis are highlighted as Assumptions 1-3. The inclusion of social sanctions is postponed to Section 5.

A given finite mass m (> 0) of risk-neutral borrowers without internal funds are endowed with one project each. The project requires an input of one unit of capital. The payoffs θ are independently and uniformly distributed: the distribution function is given by $F(\theta) = 0$ for $\theta < \underline{\theta}$, $F(\theta) = (\theta - \underline{\theta})/(\bar{\theta} - \underline{\theta})$ for $\theta \in [\underline{\theta}, \bar{\theta}]$, and $F(\theta) = 1$ for $\theta > \bar{\theta}$, where

$$\frac{\bar{\theta}}{2} > \underline{\theta} > 0.$$

MFIs offer loans. At the time a loan is made, the payoff θ is uncertain. Once realized, the project return θ is common knowledge. An individual lending (IL) contract entails a repayment r. Group lending (GL) consists of a loan of size 2 to a group of two borrowers and repayment $2r$. When the repayment decision is made, borrowers are endowed with sufficiently high income so that they are able to repay. However, the enforcement of repayment is imperfect, so borrowers choose between repaying (completely) or not (at all). The penalty for default is $p(\theta) = \theta/\beta$,[6] where

$$\beta > \max\{1, \underline{\theta}\}.^{7}$$

BC assume that the penalty consists of "two components", "a monetary loss due to seizure of income or assets" and "a non-pecuniary cost resulting from being 'hassled' by the bank, from loss of reputation, and so forth" (p. 4). Their focus on repayment rates makes an assumption with regard to the relative magnitudes of the two components dispensable. We assume that each of the two components is a constant proportion of the penalty:

Assumption 1: *Of the penalty $p(\theta)$, a fraction α $(\in [0,1])$ is pecuniary and accrues to the MFI. The remainder of the penalty is a deadweight loss.*

The two borrowers in a group ($i = 1, 2$, say) play a two-stage repayment game. At the first stage, the strategies are: contribute r to the joint repayment $2r$ or not. If both choose to contribute,

[6] There is an obvious tension between the assumptions about repayment ability on the one hand and penalties on the other hand: there is a second kind of income besides project returns which enables borrowers to repay, but the penalty is independent of the value of this income. One interpretation is that a borrower could mobilize enough money to repay by selling her belongings, but the MFI does not expect her to do this and so does not condition the penalty on the value of her belongings.

[7] This condition captures the idea that borrowers prefer the penalty over repayment in the case of minimum project return even at zero interest: $\underline{\theta}/\beta < r = 1$ (cf. BC, p. 8). Given this inequality, our former assumption $\bar{\theta}/2 > \underline{\theta}$ is implied by BC's (p. 8) somewhat stronger condition $\bar{\theta}/2 > \beta$. Without this fomer assumption, the potential advantage of GL would not materialize, and IL would be unambigously better.

the payoffs are $\theta_i - r$, where θ_i is i's realization of θ $(i = 1, 2)$. If both choose not to contribute, the payoffs are $\theta_i - p(\theta_i)$ $(i = 1, 2)$. If borrower i chooses to contribute and j $(\neq i)$ does not, i decides, at stage 2, whether she repays $2r$ alone or not. If she repays, she gets $\theta_i - 2r$ and j gets θ_j. If not, the payoffs are $\theta_1 - p(\theta_1)$ and $\theta_2 - p(\theta_2)$.

The supply of funds to the MFIs is perfectly elastic. That is, the MFIs' cost of capital is exogenously given. One may think either of private investors with a given required rate of return or of DFIs or BVFs that can do with a below market rate of return.[8] Let q denote loan supply.

Assumption 2: *The MFIs can raise any amount of capital $q \in [0, m]$ at the constant cost of capital ρ (≥ 1).*

The equilibrium contract maximizes expected borrower utility. In the case of funding by return-seeking investors, this is due to perfect competition among MFIs. For an MFI funded by a DFI, this is a natural objective.

Assumption 3: *The MFIs offer the (IL or GL) contract that maximizes borrowers' expected utility subject to the constraint that it breaks even.*

3 Repayment rates

This section gives a brief summary of BC's results on repayment probabilities with IL and GL.

Individual lending

Restrict attention to interest rates $r \geq \underline{\theta}/\beta$ (< 1), and define

$$A = [\underline{\theta}, \, \beta r).$$

Borrower i defaults if, and only if, $p(\theta_i) = \theta_i/\beta < r$, i.e., $\theta_i \in A$. So the repayment rate (i.e., the probability of repayment) is

$$\Pi_I(r) = 1 - F(\beta r) = \frac{\bar{\theta} - \beta r}{\bar{\theta} - \underline{\theta}}. \tag{1}$$

$r = \bar{\theta}/\beta$ implies a zero repayment rate: $\Pi_I(\bar{\theta}/\beta) = 0$. Evidently, the repayment rate is zero for all $r > \bar{\theta}/\beta$ as well. So without loss of generality, we can confine attention to loan rates $r \leq \bar{\theta}/\beta$.[9]

[8] Given the small proportion of microcredit markets in financial markets, an exogenous cost of capital is a natural assumption. The main results go through with an upward-sloping loan supply curve as well, and we will briefly tackle this case when we come to credit rationing.

[9] For $0 < r \leq \underline{\theta}/\beta$, the "default interval" A is not well defined, the borrower repays for all θ. Such interest rates cannot arise in equilibrium, as the MFIs' repayment falls short of ρ with certainty: $r \leq \underline{\theta}/\beta < 1 \leq \rho$.

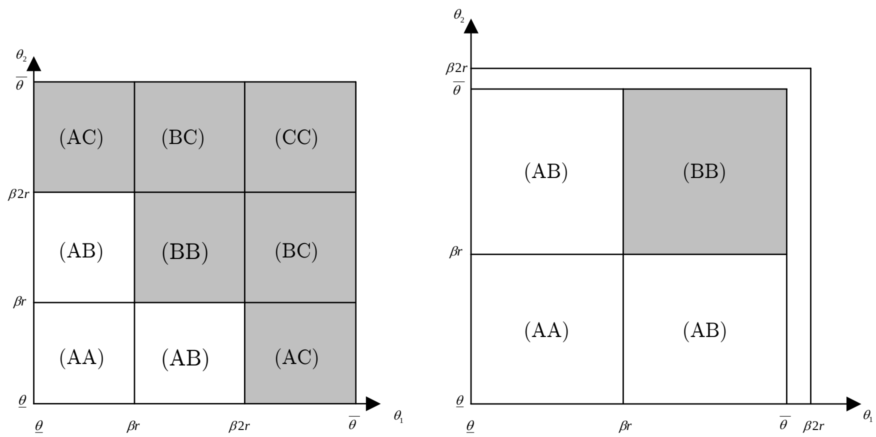

Figure 1: Cases that lead to repayment (shaded area) or default (non-shaded area) as an SPNE with GL (left panel: case L, right panel: case H)

Group lending

Again, we restrict attention to loan rates such that $\underline{\theta}/\beta \leq r \leq \bar{\theta}/\beta$.[10] By definition, $p(\theta_i) = \theta_i/\beta < r$ for $\theta_i \in A$, so borrower i prefers the penalty over repayment. We distinguish two cases: the low-interest case

$$\text{case L:} \quad \frac{\underline{\theta}}{\beta} \leq r \leq \frac{\bar{\theta}}{2\beta}$$

and the high-interest case

$$\text{case H:} \quad \frac{\bar{\theta}}{2\beta} < r \leq \frac{\bar{\theta}}{\beta}$$

(see the left and right panels of Figure 1, respectively). Let

$$B = \begin{cases} [\beta r,\ 2\beta r)\,, & \text{case L} \\ \left[\beta r,\, \bar{\theta}\right], & \text{case H} \end{cases}.$$

For $\theta_i \in B$, borrower i is willing to repay an individual loan (since $r \leq p(\theta_i) = \theta_i/\beta$) but not a group loan (since $\theta_i/\beta < 2r$). Finally, let

$$C = \left[2\beta r,\, \bar{\theta}\right], \quad \text{case L.}$$

For $\theta_i \in C$, i prefers to repay $2r$ rather than default.

BC (p. 17) characterize the subgame-perfect Nash equilibria (SPNE) of the repayment game:

(AA) For $(\theta_1, \theta_2) \in A \times A$, both players default.

[10]We shall see below (in footnote 11) that, as with IL, lenders' expected return falls short of ρ for loan rates $0 < r \leq \underline{\theta}/\beta$.

(BB) For $(\theta_1, \theta_2) \in B \times B$, both players choosing to repay is an equilibrium. Both players deciding not to repay is also an SPNE, which is however ruled out by BC (p. 7) on the grounds that it is Pareto-inferior. An alternative way to get rid of this "bad" equilibrium is elimination of weakly dominated strategies (cf. Fudenberg and Tirole, 2000, Subsection 1.1.2): the strategy not to repay at the first stage is weakly dominated by the strategy to repay.

(CC) For $(\theta_1, \theta_2) \in C \times C$ (in case L), one borrower repays $2r$ and the other free-rides.

(BC) For $(\theta_1, \theta_2) \in (B \times C) \cup (C \times B)$ (in case L), the player i with $\theta_i \in C$ repays $2r$ and the other free-rides.

In all these cases, the repayment received by the MFI is the same as with IL: $\alpha(\theta_1 + \theta_2)/\beta$ in case AA and $2r$ otherwise.

(AB) For $(\theta_1, \theta_2) \in (A \times B) \cup (B \times A)$, no-one repays. This is the drawback of GL: the borrower i with $\theta_i \in B$ would repay a single loan but is discouraged from paying back anything by joint liability.

(AC) For $(\theta_1, \theta_2) \in (A \times C) \cup (C \times A)$ (in case L), the borrower i with $\theta_i \in C$ repays $2r$. This is the advantage of GL: the high-return borrower stands in for her fellow group member.

In case L, the repayment rate is equal to the cumulated probability of cases BB, CC, BC, and AC:

$$\Pi_G(r) = 2[1 - F(2\beta r)]F(\beta r) + [1 - F(\beta r)]^2 = \frac{-3\beta^2 r^2 + 4\beta\underline{\theta} r + \bar{\theta}^2 - 2\underline{\theta}\bar{\theta}}{(\bar{\theta} - \underline{\theta})^2}, \quad \text{case L.} \tag{2}$$

In case H, the repayment rate is the probability of case BB:

$$\Pi_G(r) = [1 - F(\beta r)]^2 = \left(\frac{\bar{\theta} - \beta r}{\bar{\theta} - \underline{\theta}}\right)^2, \quad \text{case H.} \tag{3}$$

As with IL, we can confine attention to $r \leq \bar{\theta}/\beta$, because all higher interest rates imply a zero repayment rate.[11]

The BC result

BC's (p. 8) main result for the model without social sanctions is that if $\bar{\theta}/(3\beta) > 1$, then GL dominates IL in terms of repayment rates for low loan rates $r < \bar{\theta}/(3\beta)$, and vice versa. This follows from (1)-(3): $\Pi_G(r) > \Pi_I(r)$ for $r \in [1, \bar{\theta}/(3\beta))$ and $\Pi_I(r) > \Pi_G(r)$ for $r \in (\bar{\theta}/(3\beta), \bar{\theta}/\beta]$. If $\bar{\theta}/(3\beta) \leq 1$, IL yields an unambiguously higher repayment rate.

[11] Loan rates $0 < r < \underline{\theta}/\beta$ cannot occur in equilibrium with GL. The interval A is not well defined in this case, so only cases BB, BC, and CC can arise. The repayment rate is unity in each of these cases, so the MFIs are unable to break even: $r < \underline{\theta}/\beta < 1 \leq \rho$.

4 Equilibrium

This section analyzes the equilibrium of the BC model supplemented with Assumptions 1-3. We show that an equilibrium exists, consider several interesting special cases, and highlight the allocation failures that potentially arise in equilibrium.

4.1 Definition of equilibrium

Given the penalty function $p(\theta) = \theta/\beta$ and $\beta > 1$, all borrowers demand loans for any interest rate, so loan demand is constant. This is because the cost of a loan (i.e., either the interest repayment or the penalty) is less than the payoff in every state of nature: $\min\{p(\theta),\, r\} = \min\{\theta/\beta,\, r\} < \theta$.

In order to determine borrowers' expected utility, we have to make an assumption about the probability of being the borrower who repays or the free rider in case CC. The natural assumption is that each borrower has an equal chance of being the free rider:

Assumption 4: *The probability of being a borrower who repays when* $(\theta_1,\, \theta_2) \in C \times C$ *in case L under GL is* $1/2$ *for each borrower.*[12]

Denote the set of realizations of (θ_1, θ_2) that trigger default with lending type tL ($t \in \{\text{I, G}\}$) as D_t: $D_I = A$ and $D_G = (A \times A) \cup (A \times B) \cup (B \times A)$ (see the non-shaded areas in Figure 1). The complement of D_t is denoted S_t: $S_I = [\underline{\theta},\, \bar{\theta}] \backslash D_I$ and $S_G = ([\underline{\theta},\, \bar{\theta}] \times [\underline{\theta},\, \bar{\theta}]) \backslash D_G$ (see the shaded areas in Figure 1). $\Pi_I(r)$ and $\Pi_G(r)$ are the probabilities of $\theta \in S_I$ and $(\theta_1, \theta_2) \in S_G$, respectively. Let $\boldsymbol{\theta}_I = \theta$ and $\boldsymbol{\theta}_G = (\theta,\ \theta')$. Then, using Assumption 1, the MFIs' expected revenue per dollar lent with lending type tL is

$$R_t(r) = \Pi_t(r)r + [1 - \Pi_t(r)]\alpha E\left[p(\theta) \middle| \boldsymbol{\theta}_t \in D_t\right], \quad t \in \{\text{I, G}\}. \tag{4}$$

Using Assumption 4, the expected utility of a borrower who finances her project with tL is

$$U_t(r) = \Pi_t(r)E(\theta - r|\, \boldsymbol{\theta}_t \in S_t) + [1 - \Pi_t(r)]E\left[\theta - p(\theta) \middle| \boldsymbol{\theta}_t \in D_t\right], \quad t \in \{\text{I, G}\}. \tag{5}$$

We have to distinguish two types of equilibria:

Definition 1: *A lending type, a loan rate, and a quantity of loans* (tL, r, q) *are a* loan market equilibrium with market clearing *if*

[12] Since the number of borrowers who repay is equal to the number of borrowers who free-ride, the probability is necessarily 1/2 on average. Any mechanism that randomly assigns these roles to borrowers implies a probability of 1/2 for everyone.

(1) the amount of loans made is equal to demand: $q = m$*;*

(2) MFIs make zero profit: $R_t(r) = \rho$*;*

(3) no alternative contract that attracts borrowers yields positive profit: there is no $(t'L, r') \neq (tL, r)$ *such that* $R_{t'}(r') > \rho$ *and* $U_{t'}(r') \geq U_t(r)$*.*

Definition 2: *A* loan market equilibrium without trade *prevails if there is no contract that breaks even:* $R_t(r) < \rho$ *for all* (tL, r)*.*

4.2 Existence of equilibrium

Using $p(\theta) = \theta/\beta$ and (1)-(3), we obtain from (4) and (5) (see the technical appendix):

$$R_I(r) = \frac{-\beta(2-\alpha)r^2 + 2\bar{\theta}r - \frac{\alpha\underline{\theta}^2}{\beta}}{2(\bar{\theta} - \underline{\theta})} \tag{6}$$

$$U_I(r) = \frac{\beta r^2 - 2\bar{\theta}r + \bar{\theta}^2 - \left(1 - \frac{1}{\beta}\right)\underline{\theta}^2}{2(\bar{\theta} - \underline{\theta})} \tag{7}$$

for IL and

$$R_G(r) = \frac{-\beta^2(6-5\alpha)r^3 + 4\beta\underline{\theta}(2-\alpha)r^2 + 2(\bar{\theta}^2 - 2\underline{\theta}\bar{\theta} - \alpha\underline{\theta}^2)r + \frac{\alpha\underline{\theta}^3}{\beta}}{2(\bar{\theta} - \underline{\theta})^2}, \quad \text{case L}, \tag{8}$$

$$U_G(r) = \frac{\beta^2 r^3 - 4\beta\underline{\theta}r^2 + (2\underline{\theta}^2 - 2\bar{\theta}^2 + 4\underline{\theta}\bar{\theta})r + \bar{\theta}^3 - \underline{\theta}\bar{\theta}^2 - \underline{\theta}^2\bar{\theta} + \left(1 - \frac{1}{\beta}\right)\underline{\theta}^3}{2(\bar{\theta} - \underline{\theta})^2}, \quad \text{case L}, \tag{9}$$

and

$$R_G(r) = \frac{\beta^2(2-\alpha)r^3 - \beta\bar{\theta}(4-\alpha)r^2 + \bar{\theta}^2(2+\alpha)r - \frac{\alpha}{\beta}(\underline{\theta}^2\bar{\theta} + \underline{\theta}\bar{\theta}^2 - \underline{\theta}^3)}{2(\bar{\theta} - \underline{\theta})^2}, \quad \text{case H}, \tag{10}$$

$$U_G(r) = \frac{-\beta^2 r^3 + 3\beta\bar{\theta}r^2 - 3\bar{\theta}^2 r + \bar{\theta}^3 - \left(1 - \frac{1}{\beta}\right)(\underline{\theta}^2\bar{\theta} + \underline{\theta}\bar{\theta}^2 - \underline{\theta}^3)}{2(\bar{\theta} - \underline{\theta})^2}, \quad \text{case H}, \tag{11}$$

for GL. Notice that $R_G(r)$ is continuous at the border $r = \bar{\theta}/(2\beta)$ between cases L and H and that $R_I(\underline{\theta}/\beta) = R_G(\underline{\theta}/\beta) = \underline{\theta}/\beta$ and $R_I(\bar{\theta}/\beta) = R_G(\bar{\theta}/\beta) = (\alpha/\beta)(\bar{\theta} + \underline{\theta})/2$ (see the technical appendix).

Equations (6)-(11) will be used to characterize the equilibria of the types defined in Definitions 1 and 2. To pave the way for our equilibrium analysis of the BC model, we first prove existence of equilibrium:

Proposition 1: *Either a loan market equilibrium with market clearing or a loan market equilibrium without trade exists.*

Proof: For lending type t, let r_t denote the minimum interest rate in the interval $[\underline{\theta}/\beta, \bar{\theta}/\beta]$ such that $R_t(r) = \rho$ (see Figures 2 and 3 below). Since $R_t(\underline{\theta}/\beta) = \underline{\theta}/\beta < 1 \leq \rho$ and the $R_t(r)$ functions are polynomials, if $\max_{r,t} R_t(r) \geq \rho$, then r_t exists for at least one $t \in \{I, G\}$. If r_t exists for exactly one lending type tL, denote this type as t'L. If both r_I and r_G exist, let t'L be the lending type that yields higher borrower utility $U_{t'}(r_{t'})$ (if the borrower utilities are identical, pick t'L arbitrarily). We assert that $(t'\text{L}, r_{t'}, m)$ is an equilibrium with market clearing. Conditions (1) and (2) in Definition 1 are satisfied. Clearly, if $r_{t'} = \bar{\theta}/\beta$, it is not possible to raise the expected repayment beyond ρ. So consider $r_{t'} < \bar{\theta}/\beta$. By construction, $R_{t'}(r) > \rho$ requires $r > r_{t'}$. From (7), (9), and (11), $U'_t(r) < 0$ for all $r < \bar{\theta}/\beta$ and for $t \in \{I, G\}$ (see the technical appendix). So $U_{t'}(r) < U_{t'}(r_{t'})$ whenever $R_{t'}(r) > \rho$. That is, MFIs cannot make a positive profit with lending type t'. If r_t, $t \neq t'$, exists (i.e., if it is possible to break even with the other lending type as well), to make a profit $R_t(r) > \rho$ with lending type tL, MFIs must set $r > r_t$. As $U'_t(r) < 0$, this implies $U_t(r) < U_t(r_t) \leq U_{t'}(r_{t'})$. This proves the validity of (3) in Definition 1. If $\max_{r,t} R_t(r) < \rho$, from Definition 2, there is a loan market equilibrium without trade. q.e.d.

Proposition 1 ensures that an equilibrium exists for all admissible parameter values. Moreover, the proof of the proposition is constructive: equilibria with market clearing are found by looking for the minimum break-even loan rates for the two lending types and comparing the corresponding borrower expected utilities.

4.3 Special cases

There is one special case in which the BC analysis of repayment rates takes us a long way towards equilibrium, viz. $\alpha = 0$. In this special case, as the penalty $p(\theta)$ is completely non-pecuniary, (4) becomes $R_t(r) = \Pi_t(r)r$, so the lending type tL that yields the higher repayment rate at r also yields the higher expected repayment at r. However, even so, it is not straightforward to determine the optimal lending type. To see this, consider the following example:

Example 1: $\alpha = 0$, $\underline{\theta} = 1$, $\bar{\theta} = 6$, $\beta = 1.5$, $\rho = 1.02$. The minimum break-even loan rate is lower with GL than with IL: $r_G = 1.1717 < 1.2254 = r_I$ (see Figure 2). However, the associated borrower expected utilities satisfy $U_I(r_I) = 2.3214 > 2.3163 = U_G(r_G)$. Thus, the equilibrium entails IL, even though MFIs can break even with GL at a lower interest rate. Put differently, the equilibrium deadweight loss $E(\theta) - U_t(r_t) - R_t(r_t)$ caused by the non-pecuniary nature of the penalty is higher with GL (0.1637) than with IL (0.1586). To see why, note that case L prevails (since $r_G = 1.1717 < 2 = \bar{\theta}/(2\beta)$), and case AB, in which GL is disadvantageous, occurs with probability $(2F(\beta r)[F(2\beta r) - F(\beta r)] =)$ 0.1889. In that case, with GL, the expected penalty for the borrower with $\theta \in B$ is $(E(\theta|\, \theta \in B)/\beta =)$ 1.7575 – way beyond the contractual repayment. As a result, the expected penalty averaged over all borrowers is $(E(\theta|\, \boldsymbol{\theta}_G \in D_G)/\beta =)$ 1.2641

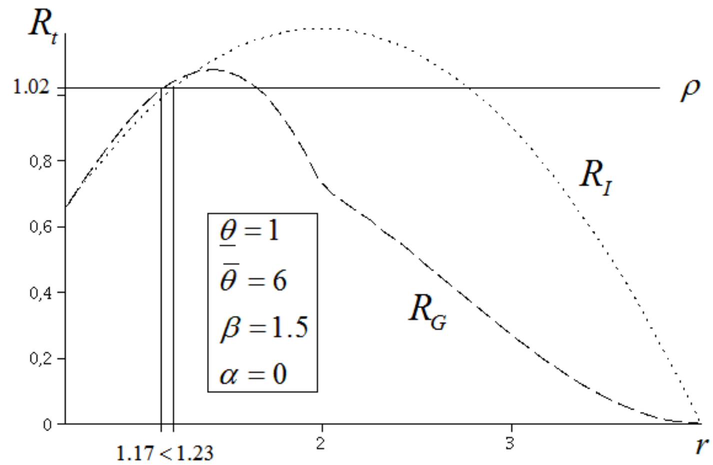

Figure 2: Example 1

and still exceeds the loan rate $r_G = 1.1717$.[13] This compares with an expected penalty of 0.9463 with IL. This example proves:

Proposition 2: *There exist parameter values such that* (IL, r_I, m) *is a loan market equilibrium with market clearing, even though* $r_G < r_I$.

Example 1 demonstrates that the fact that GL breaks even at a lower interest rate does not mean that it occurs in equilibrium. This raises the question of whether as an indicator of market outcomes repayment rates are systematically biased in favor of GL. To address this question, we pose the question: over which range of deposit rates ρ is IL the equilibrium mode of finance despite having the higher break-even interest rate? We address this question in two steps. First, we generalize Example 1. Then we conduct a systematic analysis of the parameter space.

Example 1 (ctd.): Let the parameters except ρ be as in Example 1. The maximum return that can be generated with GL is 1.0741. As can be seen from Figure 2, for all $\rho < 1.0667$, GL has the lower break-even interest rate. But the comparison of expected borrower utilities shows that GL occurs in equilibrium only for deposit rates up to $\rho = 1.0118$ (see the technical appendix). That is, for $((1.0667 - 1.0118)/(1.0667 - 1) =)$ 82.3% of the deposit rates for which GL breaks even at a lower loan rate, IL is still the equilibrium mode of finance.

Going one step further, is the high proportion of IL equilibria in instances of $r_G < r_I$ an artefact of the parameters chosen in Example 1? To investigate this issue, we consider a wide array of

[13] This inefficiency is a result of the group members' non-cooperative behavior.

model parameters. We pick $\underline{\theta}$, $\bar{\theta}$, and β from specified intervals, maintaining the assumption $\alpha = 0$. The interval of $\underline{\theta}$-values $[0.01, 1]$ is held constant. We pick $\bar{\theta}$ from one of the six intervals $[2.01, 4]$, $[2.01, 5]$, ..., $[2.01, 9]$ and β from one of the five intervals $[1.01, 1.5]$, $[1.01, 2.0]$, ..., $[1.01, 3.5]$. This gives rise to $(6 \cdot 5 =)$ 30 cases, each characterized by different upper bounds for $\bar{\theta}$ and β. In each of the 30 cases, we consider eleven different values for $\underline{\theta}$ (viz., 0.01, 0.1, 0.2, ..., 0.9, 1.0), eleven values for $\bar{\theta}$ (viz., 2.01 and ten equidistant values between 2 and the upper bound of the interval), and eleven values of β (viz., 1.01 and ten equidistant values between 1 and the upper bound of the interval). In all, this gives $(11^3 =)$ 1,331 subcases for each of the 30 cases. For each subcase, we compute the interval of deposit rates $\rho \geq 1$ (if it exists) which gives rise to $r_G < r_I$ and the subinterval for which IL is nonetheless used in equilibrium. Comparing the length of the latter subinterval with the length of the former interval gives the proportion of deposit rates with IL in equilibrium conditional on GL having the lower break-even interest rate for the given subcase (i.e, the figure comparable to the 82.3% reported for Example 1). Averaging over the subcases gives the mean proportion of instances in which IL occurs in equilibrium despite the higher interest rate for each of the 30 cases (a summary is in the technical appendix). This proportion ranges between 33.2 and 82.5 percent. The weighted average is 45.7 percent. Thus, roughly speaking, in almost half of the number of cases in which GL looks favorable judged by the break-even interest rate, IL is nonetheless the equilibrium mode of finance.

Another interesting special case is $\alpha = 1$. In this case, the penalties are 100 percent pecuniary, so there is no deadweight loss, and all zero-profit contracts are equally good from the perspective of the borrowers: $R_t(r) + U_t(r) = E(\theta)$ and $U_t(r) = E(\theta) - \rho$ when $R_t(r) = \rho$. From (6), (8), and (10), it follows that $R_G(r) > R_I(r)$ for all $\underline{\theta}/\beta < r < \bar{\theta}/\beta$ when $\alpha = 1$ (see the technical appendix). That is, the GL break-even rate r_G is lower than the IL break-even rate r_I for all ρ that give rise to loan market clearing (i.e., $\rho < (1/\beta)(\bar{\theta} + \underline{\theta})/2$). From the fact that $R_I(\underline{\theta}/\beta) = R_G(\underline{\theta}/\beta)$ and $R_I(\bar{\theta}/\beta) = R_G(\bar{\theta}/\beta)$ for all α and continuity of the functions on the right-hand sides of (6), (8), and (10) in α, it follows that for α sufficiently close to unity, if it is possible to break even with both lending types, then GL generally entails the lower break-even loan rate. One might suspect that, therefore, GL is unambiguously better than IL for α large. Interestingly, however, the assertion of Proposition 2 also holds true for α close to one, as the following example shows.

Example 2: $\alpha = 0.99$, $\underline{\theta} = 1.2$, $\bar{\theta} = 4$, $\beta = 1.5$, $\rho = 1.02$. The zero-profit loan rates are $r_G = 0.9997$ and $r_I = 1.0361$ (see Figure 3). The fact that $r_G < 1$ is interesting by itself: it

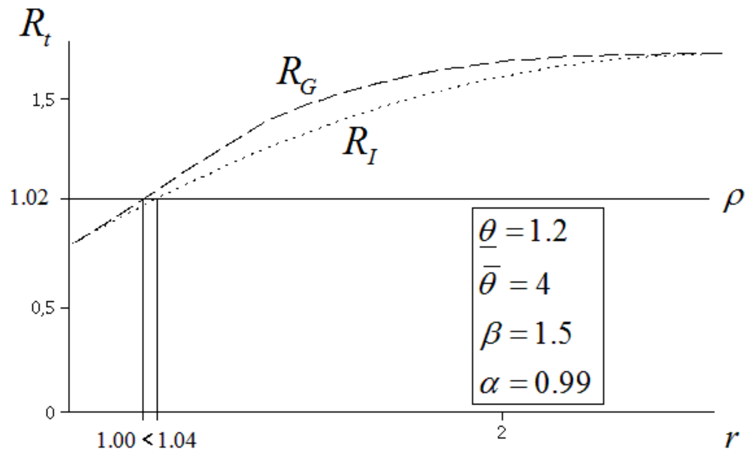

Figure 3: Example 2

shows that since (as pointed out in Example 1) the expected penalty may exceed the loan rate, a loan rate $r < 1$ may suffice to make a return $\rho > 1$. The associated expected utility levels for borrowers are $U_G = 1.5785$ and $U_I = 1.5788$.

We now turn to the allocation failures that might afflict market equilibria.

4.4 Financial fragility

There is "financial fragility" as in Mankiw (1986), in that a small increase in the cost of capital potentially induces a complete breakdown of the loan market. This follows immediately from the proof of Proposition 1: for $\max_{r,t} R_t(r) = \rho$, an equilibrium with loan market clearing exists; as soon as ρ rises, we have $\max_{r,t} R_t(r) < \rho$, and the equilibrium entails no trade. For instance, in Example 1, $\max_{r,t} R_t(r) = 1.2$, so the market collapses when the lenders require a return in excess of 20 percent. More generally, in the case $\alpha = 0$, the maximum required return beyond which the market collapses can be calculated explicitly:

Proposition 3: *Suppose $\alpha = 0$. Then*

$$\max_{r,t} R_t(r) = \frac{\bar{\theta}^2}{4\beta(\bar{\theta} - \underline{\theta})}. \tag{12}$$

When ρ rises above this value, the unique equilibrium becomes one without trade.

Proof: From (6) with $\alpha = 0$, $R_I(r)$ attains its maximum at $r = \bar{\theta}/(2\beta)$, and the maximum value is given by the right-hand side of (12). It can be shown that the functions on the right-hand sides of (8) and (10) fall short of this value in cases L and H, respectively (see the technical appendix). q.e.d.

The final two paragraphs of this section deal with two slight variations of the model, which give rise to redlining and rationing.

4.5 Redlining

Assume there is a (non-empty) finite set J of observationally distinguishable classes of borrowers of the type introduced in Section 2. Parameters, variables, and functions referring to class $j \in J$ are distinguished by a superscript j. For instance, $R_t^j(r)$ gives the return on lending at a loan rate r with lending type t to type-j borrowers. Let $m = \sum_{j \in J} m^j$, so that Assumption 2 states that the supply of capital at ρ is sufficient to finance all projects of all classes. An equilibrium prevails if for each type $j \in J$, the conditions of either Definition 1 or Definition 2 are satisfied. *Redlining* is said to prevail when there is market clearing for some types j and no trade for others in equilibrium.

Proposition 4: *Redlining prevails if, and only if,*

$$\min_j \max_{r,t} R_t^j(r) < \rho \leq \max_j \max_{r,t} R_t^j(r).$$

Proof: The equilibrium lending type, interest rate, and loan volume for each class are found following the same steps as in the proof of Proposition 1. Types j with $\rho \leq \max_{r,t} R_t^j(r)$ get a loan (via IL or GL), types j with $\max_{r,t} R_t^j(r) < \rho$ do not. So the condition in the theorem ensures that some classes get loans, while others do not. q.e.d.

The most interesting case arises when classes differ not with regard to payoffs but with regard to the nature and magnitude of the penalties: $\underline{\theta}^j = \underline{\theta}$ and $\bar{\theta}^j = \bar{\theta}$ for all j, but the α^j's and/or β^j's differ. In this case, if the condition of the proposition is satisfied, some borrowers do not get credit, even though others with equally good projects do.

Example 3: There are three classes: $J = \{1, 2, 3\}$. Penalties are non-pecuniary, the cost of capital and the payoffs are as in Example 1: $\rho = 1.02$, $\alpha^j = 0$, $\bar{\theta}^j = 6$, and $\underline{\theta}^j = 1$ for all j. The penalty parameters are $\beta^1 = 1.25$, $\beta^2 = 1.5$, and $\beta^3 = 2$. Class-1 borrowers get loans with GL at $r_G^1 = 1.0613$. As seen in Example 2, borrowers of type $j = 2$ get individual loans at $r_I^2 = 1.2254$. For class 3, there is no way to break even: $\max_{r,t} R_t^3(r) = 0.9$. Due to the limited scope for punishment after non-repayment, borrowers in this class are redlined.

The model with observationally distinguishable borrower classes has an important implication for the roles of for-profit organizations and of DFIs, which can do with lower reruns: to maximize

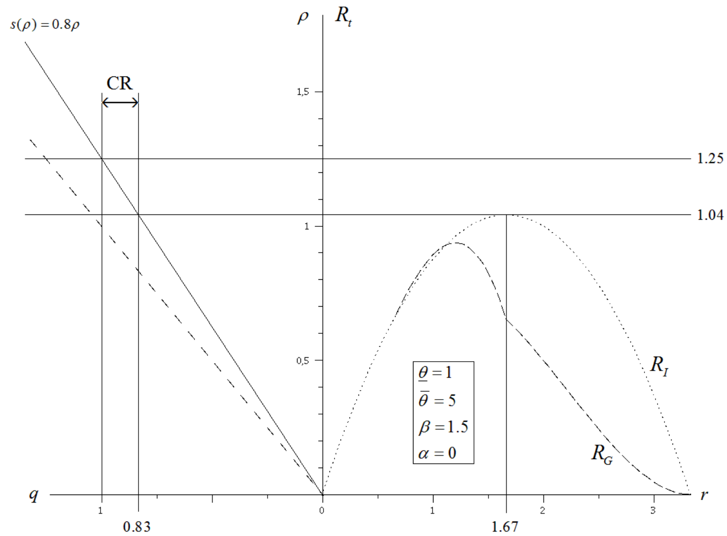

Figure 4: Example 4 (credit rationing)

the amount of credit given, DFIs have to direct their funds under management to banks that finance classes with maximum expected repayment just above their required gross return, while private investors target the high-yield market segments, for otherwise the DFIs would crowd out private investment.

4.6 Credit rationing

Going back to the one-class case, assume now, instead of Assumption 2, that the loan supply is a real-valued, strictly increasing function $s(\rho)$. A lending type, a loan rate, and a quantity of loans $(t\mathrm{L}, r, q)$ are a *credit rationing equilibrium* if (1) $0 < q < m$, (2) and (3) in Definition 1 are satisfied, and (4) $q = s(\rho)$. A credit rationing equilibrium occurs when the supply of funds at the return-maximizing interest rate falls short of the demand for credit:

Proposition 5: *If*

$$s\left[\max_{r,t} R_t(r)\right] < m,$$

then a credit rationing equilibrium occurs.

Example 4: Let $\alpha = 0$, $\underline{\theta} = 1$, $\bar{\theta} = 5$, and $\beta = 1.5$. Further, let $m = 1$ and $s(\rho) = 0.8\rho$. The maximum lender return $\max_{r,t} R_t(r) = 1.0417$ is obtained with IL at $r = 1.6667$. The corresponding loan supply is $s(1.2) = 0.8333$. So in equilibrium, MFIs make individual loans to 83.33 percent of the borrowers at $r = 1.6667$ (see Figure 4). The projects' expected return

		player 2	
		repay	don't
player 1	repay	$\theta_1 - r \,/\, \theta_2 - r$	$\theta_1 - \frac{\theta_1}{\beta} \,/\, \theta_2 - \frac{\theta_2}{\beta} - s$
	don't	$\theta_1 - \frac{\theta_1}{\beta} \,/\, \theta_2 - \frac{\theta_2}{\beta}$	$\theta_1 - \frac{\theta_1}{\beta} \,/\, \theta_2 - \frac{\theta_2}{\beta}$

Figure 5: Stage 1 of the repayment game with social sanctions: case AB ($\theta_1 \in B$, $\theta_2 \in A$)

(i.e., $(\bar{\theta} + \underline{\theta})/2 = 3$) is way beyond the level needed to induce savers to supply enough capital to finance all projects (i.e., $\rho = 1.25$), but the enforcement problem leads to credit rationing.

5 Social sanctions

Following BC (Section 4), we introduce social sanctions to the model. BC's main result in this regard is that if social sanctions are severe enough, GL yields a higher repayment rate than IL (BC, Proposition 3, p. 12). We adopt a simple specification of social sanctions and show that even if they ensure that GL implies the higher repayment rate, the loan market equilibrium still displays the allocation problems analyzed in Subsections 4.4-4.6.

Assumption 5: *If a borrower i in a group is willing to repay an individual loan (i.e., $\theta_i \geq \beta r$) and decides to repay at stage 1 of the repayment game, then if her fellow group member j decides not to repay, i imposes a sanction $s > r$ on her. No sanctions are imposed otherwise.*

That is, a social sanction is imposed when one borrower's decision not to repay forces her fellow group member to choose between repaying the group loan alone or accepting the penalty despite her declared willingness to repay.[14] As to the severity of the sanctions, since, as before, we can focus on interest rates $r \leq \bar{\theta}/\beta$, we could alternatively assume $s > \bar{\theta}/\beta$, so that the sanction simply has to be "sufficiently large" relative to model parameters, irrespective of its specific dependence on r, θ_i, and θ_j.[15]

The presence of the sanction strengthens the incentives to contribute in the repayment game. For the cases defined in Section 3, the following SPNE arise:

[14] The assumption that no sanctions are imposed otherwise is immaterial. Adding sanctions in other instances as well strengthens our conclusions. For the sake of clarity of exposition, we choose just the minimal set of sanctions that make GL become the dominant mode of finance.

[15] In particular, the sanction may or may not differ depending on whether i repays $2r$ or accepts $p(\theta_i)$ at stage 2.

(AA) Social sanctions do not play a role in this case, because neither player would repay an individual loan. Both players default.

(AB) This is the critical case for GL. Repaying is a weakly dominant strategy for both players at stage 1. So repayment becomes an SPNE (see Figure 5). Both players choosing not to repay is also an SPNE. As in Section 3, we rule out this SPNE because it requires that both play a weakly dominated strategy.[16]

(AC) The unique SPNE entails that both players repay.

(BB) Both players choosing to repay is an equilibrium. The other SPNE, in which both players decide not to repay, is ruled out on the grounds that it requires that one borrower plays a weakly dominated strategy and is Pareto-inferior.

(BC) Repaying is a strictly dominant strategy for the player i with $\theta_i \in C$ and a weakly dominant strategy for the other player. The unique SPNE entails repayment.

(CC) Repaying is a strictly dominant strategy at stage 1 for both players. Neither tries to free-ride.

Assumption 5 thus eliminates the drawback of GL: repayment occurs unless case AA occurs. The repayment rate becomes

$$\Pi_G(r) = 1 - F(\beta r)^2 = \frac{-\beta^2 r^2 + 2\beta\underline{\theta}r + \bar{\theta}^2 - 2\underline{\theta}\bar{\theta}}{(\bar{\theta} - \underline{\theta})^2} \tag{13}$$

(there is no need to distinguish cases L and H). GL dominates IL in that it brings about repayment in case AC, when the borrower i with $\theta_i \in C$ stands in for her fellow group member. Accordingly, from (1) and (13), $\Pi_G(r) > \Pi_I(r)$ whenever $F(\beta r) < 1$, i.e., $r < \bar{\theta}/\beta$. As before, we can confine attention to $r \leq \bar{\theta}/\beta$ because $\Pi_t(r) = 0$ for $r > \bar{\theta}/\beta$ ($t \in \{I, G\}$). For the sake of convenience, we also restrict attention to the case $\alpha = 0$, so that $R_G(r) = \Pi_G(r)r > \Pi_I(r)r = R_I(r)$ for $r < \bar{\theta}/\beta$. The function $R_G(r)$ has the characteristic hump shape over the interval $(\underline{\theta}/\beta,\ \bar{\theta}/\beta)$ (see the technical appendix). Letting r_G denote the minimum interest rate that helps MFIs break even, we have:

Proposition 6: *Let $\alpha = 0$. If $\max_r R_G(r) \geq \rho$, the unique equilibrium is (GL, r_G, m). Otherwise the unique equilibrium entails no trade.*

Proof: Suppose $\max_r R_G(r) \geq \rho$, so that r_G is well defined. As in the absence of social sanctions, GL contracts with higher interest rates yield lower expected borrower utility $U_G(r)$ (see the technical appendix). So it is not possible to make a profit with a different GL contract. Hence, (GL, r_G, m) is the unique equilibrium if $\max_r R_I(r) < \rho$. In case it is possible to break even

[16] The outcome is not Pareto-inferior compared with the former SPNE.

with IL as well, let r_I denote the lowest break-even interest rate. The fact that $R_G(r) > R_I(r)$ for all r implies that $r_I > r_G$. It follows that the deadweight loss with GL at r_I (i.e., $E(\theta) - U_G(r_I) - R_G(r_I)$) is higher than at r_G (see the technical appendix). Furthermore, IL causes a higher deadweight loss than GL at each given interest rate (see the technical appendix), so that $U_I(r_I) < U_G(r_G)$. That is, borrowers prefer GL at r_G to IL at r_I. Finally, an increase in the interest rate raises the deadweight loss of IL, so that $U_I(r) < U_G(r_G)$ for all interest rates r that help banks to break even with IL. Hence, it is not possible to enter the market with a profitable IL contract. Clearly, no trade is the only viable equilibrium when $\max_r R_G(r) < \rho$. q.e.d

Proposition 6 shows that the disadvantage of GL, which potentially makes IL the equilibrium mode of finance despite the higher break-even interest rate (cf. Proposition 2), can be overcome by means of social sanctions: with social sanctions obeying Assumption 5, lending takes place using a GL contract whenever the volume of trade is positive in equilibrium. This does not mean, however, that GL helps get rid of the enforcement problem altogether: the fact that the expected repayment $R_G(r)$ is hump-shaped implies that the allocation problems encountered in Subsections 4.4-4.6 continue to be prevalent. That is, there is financial fragility, in that the market collapses when ρ rises beyond $\max_r R_G(r)$; if there are several borrower classes j, those with $\max_r R_G^j(r) < \rho$ are redlined; and if the capital supply is a strictly increasing function $s(\rho)$, credit rationing arises if $s[\max_r R_G(r)] < m$ (cf. Propositions 3-5).

6 Conclusion

BC analyze repayment rates in a GL model with enforcement problems. The recent trend towards private investments and market instruments in microfinance markets raises the question of what equilibrium in the BC model looks like. The present paper shows that, no matter whether social sanctions are present or not and irrespective of the type of finance used, the market equilibrium suffers from the usual allocation problems known from the imperfect information literature. This means that the prospective growth of the market for microcredit is unlikely to be a frictionless process. MFIs will have to continue to take due care that borrowers have proper incentives to repay. If the DFIs' objective is to maximize the loan volume, they should target MFIs active in the less profitable segments of the market and leave the more profitable business to private investors.

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Technical Appendix

Expected repayment and expected utility in case L:

$$
\begin{aligned}
E(\theta|\,\boldsymbol{\theta}_G \in D_G) &= \frac{1}{1-\Pi_G(r)}\left(\int_{\underline{\theta}}^{2\beta r}\frac{1}{\bar{\theta}-\underline{\theta}}\int_{\underline{\theta}}^{\beta r}\frac{\theta}{\bar{\theta}-\underline{\theta}}d\theta\,d\theta' + \int_{\underline{\theta}}^{\beta r}\frac{1}{\bar{\theta}-\underline{\theta}}\int_{\beta r}^{2\beta r}\frac{\theta}{\bar{\theta}-\underline{\theta}}d\theta\,d\theta'\right)\\
&= \frac{1}{1-\Pi_G(r)}\left[\frac{F(2\beta r)}{\bar{\theta}-\underline{\theta}}\int_{\underline{\theta}}^{\beta r}\theta\,d\theta + \frac{F(\beta r)}{\bar{\theta}-\underline{\theta}}\int_{\beta r}^{2\beta r}\theta\,d\theta\right]\\
&= \frac{1}{1-\Pi_G(r)}\left[\frac{2\beta r-\underline{\theta}}{(\bar{\theta}-\underline{\theta})^2}\frac{\beta^2 r^2-\underline{\theta}^2}{2} + \frac{\beta r-\underline{\theta}}{(\bar{\theta}-\underline{\theta})^2}\frac{4\beta^2 r^2-\underline{\theta}^2}{2}\right]\\
&= \frac{1}{1-\Pi_G(r)}\frac{5\beta^3 r^3-4\beta^2\underline{\theta}r^2-2\beta\underline{\theta}^2 r+\underline{\theta}^3}{2(\bar{\theta}-\underline{\theta})^2}\\
R_G(r) &= \Pi_G(r)r + [1-\Pi_G(r)]\alpha E\left(\frac{\theta}{\beta}\middle|\boldsymbol{\theta}_G \in D_G\right)\\
&= \frac{1}{2(\bar{\theta}-\underline{\theta})^2}\left[(-6\beta^2 r^2+8\beta\underline{\theta}r+2\bar{\theta}^2-4\underline{\theta}\bar{\theta})r + \frac{\alpha}{\beta}(5\beta^3 r^3-4\beta^2\underline{\theta}r^2-2\beta\underline{\theta}^2 r+\underline{\theta}^3)\right]\\
&= \frac{-\beta^2(6-5\alpha)r^3+4\beta\underline{\theta}(2-\alpha)r^2+2(\bar{\theta}^2-2\underline{\theta}\bar{\theta}-\alpha\underline{\theta}^2)r+\frac{\alpha\underline{\theta}^3}{\beta}}{2(\bar{\theta}-\underline{\theta})^2}.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\bar{\theta}+\underline{\theta}}{2} &= E(\theta)\\
&= \Pi_G(r)E(\theta|\,\boldsymbol{\theta}_G \in S_G) + [1-\Pi_G(r)]E(\theta|\,\boldsymbol{\theta}_G \in D_G)\\
\Pi_G(r)E(\theta|\,\boldsymbol{\theta}_G \in S_G) &= \frac{\bar{\theta}+\underline{\theta}}{2} - [1-\Pi_G(r)]E(\theta|\,\boldsymbol{\theta}_G \in D_G)\\
&= \frac{\bar{\theta}+\underline{\theta}}{2} - \frac{5\beta^3 r^3-4\beta^2\underline{\theta}r^2-2\beta\underline{\theta}^2 r+\underline{\theta}^3}{2(\bar{\theta}-\underline{\theta})^2}\\
&= \frac{-5\beta^3 r^3+4\beta^2\underline{\theta}r^2+2\beta\underline{\theta}^2 r+\bar{\theta}^3-\underline{\theta}\bar{\theta}^2-\underline{\theta}^2\bar{\theta}}{2(\bar{\theta}-\underline{\theta})^2}\\
U_G(r) &= \Pi_G(r)E(\theta-r|\,\boldsymbol{\theta}_G \in S_G) + [1-\Pi_G(r)]E\left(\theta-\frac{\theta}{\beta}\middle|\boldsymbol{\theta}_G \in D_G\right)\\
&= \frac{-5\beta^3 r^3+4\beta^2\underline{\theta}r^2+2\beta\underline{\theta}^2 r+\bar{\theta}^3-\underline{\theta}\bar{\theta}^2-\underline{\theta}^2\bar{\theta}}{2(\bar{\theta}-\underline{\theta})^2}\\
&\quad -\frac{-3\beta^2 r^2+4\beta\underline{\theta}r+\bar{\theta}^2-2\underline{\theta}\bar{\theta}}{(\bar{\theta}-\underline{\theta})^2}r\\
&\quad +\left(1-\frac{1}{\beta}\right)\frac{5\beta^3 r^3-4\beta^2\underline{\theta}r^2-2\beta\underline{\theta}^2 r+\underline{\theta}^3}{2(\bar{\theta}-\underline{\theta})^2}\\
&= \frac{\beta^2 r^3-4\beta\underline{\theta}r^2+(2\underline{\theta}^2-2\bar{\theta}^2+4\underline{\theta}\bar{\theta})r+\bar{\theta}^3-\underline{\theta}\bar{\theta}^2-\underline{\theta}^2\bar{\theta}+\left(1-\frac{1}{\beta}\right)\underline{\theta}^3}{2(\bar{\theta}-\underline{\theta})^2}.
\end{aligned}
$$

Expected repayment and expected utility in case H:

$$\begin{aligned}
E(\theta|\,\boldsymbol{\theta}_G \in D_G) &= \frac{1}{1-\Pi_G(r)}\left(\int_{\underline{\theta}}^{\bar{\theta}} \frac{1}{\bar{\theta}-\underline{\theta}} \int_{\underline{\theta}}^{\beta r} \frac{\theta}{\bar{\theta}-\underline{\theta}} d\theta\, d\theta' + \int_{\underline{\theta}}^{\beta r} \frac{1}{\bar{\theta}-\underline{\theta}} \int_{\beta r}^{\bar{\theta}} \frac{\theta}{\bar{\theta}-\underline{\theta}} d\theta\, d\theta'\right) \\
&= \frac{1}{1-\Pi_G(r)}\left[\frac{1}{\bar{\theta}-\underline{\theta}} \int_{\underline{\theta}}^{\beta r} \theta\, d\theta + \frac{F(\beta r)}{\bar{\theta}-\underline{\theta}} \int_{\beta r}^{\bar{\theta}} \theta\, d\theta\right] \\
&= \frac{1}{1-\Pi_G(r)}\left[\frac{1}{\bar{\theta}-\underline{\theta}} \frac{\beta^2 r^2 - \underline{\theta}^2}{2} + \frac{\beta r - \underline{\theta}}{(\bar{\theta}-\underline{\theta})^2} \frac{\bar{\theta}^2 - \beta^2 r^2}{2}\right] \\
&= \frac{1}{1-\Pi_G(r)} \frac{-\beta^3 r^3 + \beta^2 \bar{\theta} r^2 + \beta \bar{\theta}^2 r - \underline{\theta}^2 \bar{\theta} - \underline{\theta}\bar{\theta}^2 + \underline{\theta}^3}{2(\bar{\theta}-\underline{\theta})^2} \\
R_G(r) &= \Pi_G(r) r + [1-\Pi_G(r)]\alpha E\left(\frac{\theta}{\beta}\middle|\boldsymbol{\theta}_G \in D_G\right) \\
&= \left(\frac{\bar{\theta}-\beta r}{\bar{\theta}-\underline{\theta}}\right)^2 r + \frac{\alpha}{\beta} \frac{-\beta^3 r^3 + \beta^2 \bar{\theta} r^2 + \beta \bar{\theta}^2 r - \underline{\theta}^2 \bar{\theta} - \underline{\theta}\bar{\theta}^2 + \underline{\theta}^3}{2(\bar{\theta}-\underline{\theta})^2} \\
&= \frac{\beta^2(2-\alpha) r^3 - \beta\bar{\theta}(4-\alpha) r^2 + \bar{\theta}^2(2+\alpha) r - \frac{\alpha}{\beta}(\underline{\theta}^2\bar{\theta} + \underline{\theta}\bar{\theta}^2 - \underline{\theta}^3)}{2(\bar{\theta}-\underline{\theta})^2}.
\end{aligned}$$

$$\begin{aligned}
\Pi_G(r)E(\theta|\,\boldsymbol{\theta}_G \in S_G) &= \frac{\bar{\theta}+\underline{\theta}}{2} - [1-\Pi_G(r)]E(\theta|\,\boldsymbol{\theta}_G \in D_G) \\
&= \frac{\bar{\theta}+\underline{\theta}}{2} - \frac{-\beta^3 r^3 + \beta^2 \bar{\theta} r^2 + \beta \bar{\theta}^2 r - \underline{\theta}^2 \bar{\theta} - \underline{\theta}\bar{\theta}^2 + \underline{\theta}^3}{2(\bar{\theta}-\underline{\theta})^2} \\
&= \frac{\beta^3 r^3 - \beta^2 \bar{\theta} - \beta\bar{\theta}^2 r + \bar{\theta}^3}{2(\bar{\theta}-\underline{\theta})^2} \\
U_G(r) &= \Pi_G(r)E(\theta - r|\,\boldsymbol{\theta}_G \in S_G) + [1-\Pi_G(r)]E\left(\theta - \frac{\theta}{\beta}\middle|\boldsymbol{\theta}_G \in D_G\right) \\
&= \frac{\beta^3 r^3 - \beta^2 \bar{\theta} r^2 - \beta\bar{\theta}^2 r + \bar{\theta}^3}{2(\bar{\theta}-\underline{\theta})^2} - \left(\frac{\bar{\theta}-\beta r}{\bar{\theta}-\underline{\theta}}\right)^2 r \\
&\quad + \left(1-\frac{1}{\beta}\right)\frac{-\beta^3 r^3 + \beta^2 \bar{\theta} r^2 + \beta \bar{\theta}^2 r - \underline{\theta}^2 \bar{\theta} - \underline{\theta}\bar{\theta} + \underline{\theta}^3}{2(\bar{\theta}-\underline{\theta})^2} \\
&= \frac{\bar{\theta}^3}{2(\bar{\theta}-\underline{\theta})^2} - \frac{2\bar{\theta}^2 r - 4\beta\bar{\theta} r^2 + 2\beta^2 r^3}{2(\bar{\theta}-\underline{\theta})^2} \\
&\quad + \frac{-\underline{\theta}^2\bar{\theta} - \underline{\theta}\bar{\theta}^2 + \underline{\theta}^3 + \beta^2 r^3 - \beta\bar{\theta} r^2 - \bar{\theta}^2 r + \frac{1}{\beta}(\underline{\theta}^2\bar{\theta} + \underline{\theta}\bar{\theta}^2 - \underline{\theta}^3)}{2(\bar{\theta}-\underline{\theta})^2} \\
&= \frac{-\beta^2 r^3 + 3\beta\bar{\theta} r^2 - 3\bar{\theta}^2 r + \bar{\theta}^3 - \left(1-\frac{1}{\beta}\right)(\underline{\theta}^2\bar{\theta} + \underline{\theta}\bar{\theta}^2 - \underline{\theta}^3)}{2(\bar{\theta}-\underline{\theta})^2}.
\end{aligned}$$

Proof that $R_I(\underline{\theta}/\beta) = R_G(\underline{\theta}/\beta) = \underline{\theta}/\beta$:

$$
\begin{aligned}
R_I\left(\frac{\underline{\theta}}{\beta}\right) &= \frac{-\frac{\underline{\theta}^2}{\beta} + 2\frac{\underline{\theta}\bar{\theta}}{\beta} - \frac{\underline{\theta}^2}{\beta}}{2(\bar{\theta}-\underline{\theta})} \\
&= \frac{\frac{\underline{\theta}}{\beta}(\bar{\theta}-\underline{\theta})}{\bar{\theta}-\underline{\theta}} \\
&= \frac{\underline{\theta}}{\beta} \\
R_G\left(\frac{\underline{\theta}}{\beta}\right) &= \frac{-\beta^2(6-5\alpha)\frac{\underline{\theta}^3}{\beta^3} + 4\beta\underline{\theta}(2-\alpha)\frac{\underline{\theta}^2}{\beta^2} + 2(\bar{\theta}^2 - 2\underline{\theta}\bar{\theta} - \alpha\underline{\theta}^2)\frac{\underline{\theta}}{\beta} + \alpha\frac{\underline{\theta}^3}{\beta}}{2(\bar{\theta}-\underline{\theta})^2} \\
&= \frac{2\frac{\underline{\theta}^3}{\beta} + 2\frac{\underline{\theta}}{\beta}(\bar{\theta}^2 - 2\underline{\theta}\bar{\theta})}{2(\bar{\theta}-\underline{\theta})^2} \\
&= \frac{\underline{\theta}}{\beta}\frac{\underline{\theta}^2 - 2\underline{\theta}\bar{\theta} + \bar{\theta}^2}{(\bar{\theta}-\underline{\theta})^2} \\
&= \frac{\underline{\theta}}{\beta}.
\end{aligned}
$$

Proof that $R_I(\bar{\theta}/\beta) = R_G(\bar{\theta}/\beta) = (\alpha/\beta)(\bar{\theta}+\underline{\theta})/2$:

$$
\begin{aligned}
R_I\left(\frac{\bar{\theta}}{\beta}\right) &= \frac{-\beta(2-\alpha)\left(\frac{\bar{\theta}}{\beta}\right)^2 + 2\bar{\theta}\frac{\bar{\theta}}{\beta} - \alpha\frac{\underline{\theta}^2}{\beta}}{2(\bar{\theta}-\underline{\theta})} \\
&= \frac{\alpha\frac{\bar{\theta}^2}{\beta} - \alpha\frac{\underline{\theta}^2}{\beta}}{2(\bar{\theta}-\underline{\theta})} \\
&= \frac{\alpha}{\beta}\frac{\bar{\theta}+\underline{\theta}}{2} \\
R_G\left(\frac{\bar{\theta}}{\beta}\right) &= \frac{\beta^2(2-\alpha)\left(\frac{\bar{\theta}}{\beta}\right)^3 + \beta\bar{\theta}(4-\alpha)\left(\frac{\bar{\theta}}{\beta}\right)^2 + \bar{\theta}^2(2+\alpha)\frac{\bar{\theta}}{\beta} - \frac{\alpha}{\beta}(\underline{\theta}^2\bar{\theta} + \underline{\theta}\bar{\theta}^2 - \underline{\theta}^3)}{2(\bar{\theta}-\underline{\theta})^2} \\
&= \frac{(2-\alpha)\frac{\bar{\theta}^3}{\beta} - (4-\alpha)\frac{\bar{\theta}^3}{\beta} + (2+\alpha)\frac{\bar{\theta}^3}{\beta} - \frac{\alpha}{\beta}(\underline{\theta}^2\bar{\theta} + \underline{\theta}\bar{\theta}^2 - \underline{\theta}^3)}{2(\bar{\theta}-\underline{\theta})^2} \\
&= \frac{\alpha}{\beta}\frac{\bar{\theta}^3 - \underline{\theta}^2\bar{\theta} - \underline{\theta}\bar{\theta}^2 + \underline{\theta}^3}{2(\bar{\theta}-\underline{\theta})^2} \\
&= \frac{\alpha}{\beta}\frac{\bar{\theta}^2(\bar{\theta}-\underline{\theta}) - \underline{\theta}^2(\bar{\theta}-\underline{\theta})}{2(\bar{\theta}-\underline{\theta})^2} \\
&= \frac{\alpha}{\beta}\frac{\bar{\theta}^2 - \underline{\theta}^2}{2(\bar{\theta}-\underline{\theta})} \\
&= \frac{\alpha}{\beta}\frac{\bar{\theta}+\underline{\theta}}{2}.
\end{aligned}
$$

Proof that $U_t'(r) < 0$ for $r < \bar{\theta}/\beta$, $t \in \{I, G\}$:

Individual lending

From (7),

$$
\begin{aligned}
U_I'(r)(\bar{\theta} - \underline{\theta}) &= \beta r - \underline{\theta} \\
&< 0 \Leftrightarrow \\
r &< \frac{\bar{\theta}}{\beta}.
\end{aligned}
$$

Group lending, case L

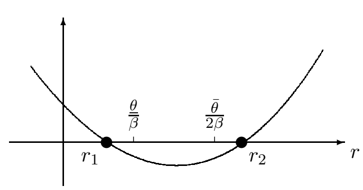
Let $U_{GL}(r)$ denote the function on the right-hand side of (9). Differentiating with respect to r shows gives

$$
\begin{aligned}
U_{GL}'(r)2(\bar{\theta} - \underline{\theta})^2 &= 3\beta^2 r^2 - 8\beta\underline{\theta} r + (2\underline{\theta}^2 - 2\bar{\theta}^2 + 4\underline{\theta}\bar{\theta}) \\
&= 3\beta^2 \left(r^2 - \frac{8}{3}\frac{\underline{\theta}}{\beta} r + \frac{2}{3}\frac{\underline{\theta}^2 - \bar{\theta}^2 + 2\underline{\theta}\bar{\theta}}{\beta^2} \right) \\
&< 0 \Leftrightarrow \\
0 &> r^2 - \frac{8}{3}\frac{\underline{\theta}}{\beta} r + \frac{2}{3}\frac{\underline{\theta}^2 - \bar{\theta}^2 + 2\underline{\theta}\bar{\theta}}{\beta^2}.
\end{aligned}
$$

Let $x \equiv \bar{\theta}/(2\underline{\theta}) > 1$. The roots of the equation of the right-hand side of the inequality are

$$
\begin{aligned}
r_{1/2} &= \frac{4}{3}\frac{\underline{\theta}}{\beta} \pm \left\{ \frac{16}{9}\left(\frac{\underline{\theta}}{\beta}\right)^2 - \frac{2}{3}\left(\frac{\underline{\theta}}{\beta}\right)^2 \left[1 - \left(\frac{\bar{\theta}}{\underline{\theta}}\right)^2 + 2\frac{\bar{\theta}}{\underline{\theta}} \right] \right\}^{\frac{1}{2}} \\
&= \frac{4}{3}\frac{\underline{\theta}}{\beta} \left[1 \pm \left(\frac{3}{2}x^2 - \frac{3}{2}x + \frac{5}{8} \right)^{\frac{1}{2}} \right].
\end{aligned}
$$

Since $x > 1$, the discriminant is positive, so there are two real roots.



The smaller root r_1 is less than $\underline{\theta}/\beta$ iff

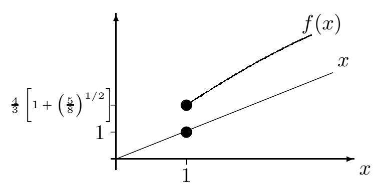
$$
\begin{aligned}
\frac{\underline{\theta}}{\beta} &> r_1 \\
&= \frac{4}{3}\frac{\underline{\theta}}{\beta} \left[1 - \left(\frac{3}{2}x^2 - \frac{3}{2}x + \frac{5}{8} \right)^{\frac{1}{2}} \right] \\
1 &> \frac{4}{3} \left[1 - \left(\frac{3}{2}x^2 - \frac{3}{2}x + \frac{5}{8} \right)^{\frac{1}{2}} \right].
\end{aligned}
$$

Since $x > 1$, a sufficient condition for the validity of this inequality is

$$\begin{aligned} 1 &> \frac{4}{3}\left[1 - \left(\frac{5}{8}\right)^{\frac{1}{2}}\right] \\ &= 0.2792. \end{aligned}$$

The bigger root r_2 is greater than $\bar{\theta}/(2\beta)$ iff

$$\begin{aligned} \frac{\bar{\theta}}{2\beta} &< r_2 \\ &= \frac{4}{3}\frac{\underline{\theta}}{\beta}\left[1 + \left(\frac{3}{2}x^2 - \frac{3}{2}x + \frac{5}{8}\right)^{\frac{1}{2}}\right] \\ x &< \frac{4}{3}\left[1 + \left(\frac{3}{2}x^2 - \frac{3}{2}x + \frac{5}{8}\right)^{\frac{1}{2}}\right] \\ &\equiv f(x). \end{aligned}$$



This follows from the fact that

$$f(1) = \frac{4}{3}\left[1 + \left(\frac{5}{8}\right)^{\frac{1}{2}}\right] > 1$$

and $f'(x) > 1$ for all $x \geq 1$:

$$\begin{aligned} f'(x) &= \frac{4}{3}\left(3x - \frac{3}{2}\right)\frac{1}{2}\left(\frac{3}{2}x^2 - \frac{3}{2}x + \frac{5}{8}\right)^{-\frac{1}{2}} \\ &> 1 \Leftrightarrow \\ 12x - 6 &> 6\left(\frac{3}{2}x^2 - \frac{3}{2}x + \frac{5}{8}\right)^{\frac{1}{2}} \\ 2x - 1 &> \left(\frac{3}{2}x^2 - \frac{3}{2}x + \frac{5}{8}\right)^{\frac{1}{2}} \\ 4x^2 - 4x + 1 &> \frac{3}{2}x^2 - \frac{3}{2}x + \frac{5}{8} \\ \frac{5}{2}x(x-1) + \frac{3}{8} &> 0. \end{aligned}$$

So the r-values consistent with the definition of case L form a subset of the r-values such that $U'_{GL}(r) < 0$:

$$\left[\frac{\underline{\theta}}{\beta}, \frac{\bar{\theta}}{2\beta}\right] \subset (r_1, r_2).$$

This proves $U'_G(r) < 0$ in case L.

Group lending, case H

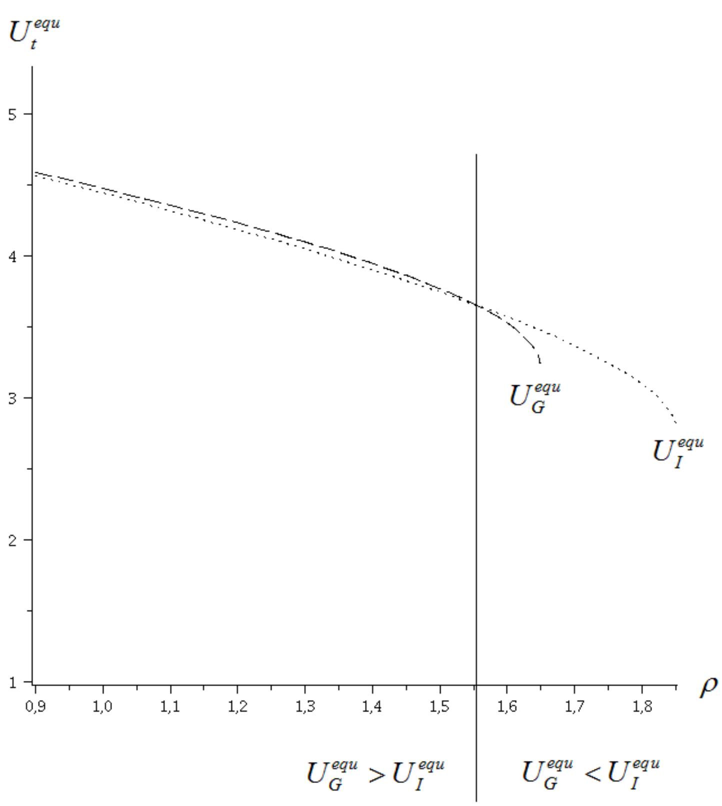
Letting $U_{GH}(r)$ denote the function on the right-hand side of (11), we have:

$$\begin{aligned} U'_{GH}(r)2(\bar{\theta}-\underline{\theta})^2 &= -3\beta^2 r^2 + 6\beta\bar{\theta}r - 3\bar{\theta}^2 \\ &= -3\beta^2\left[r^2 - 2\frac{\bar{\theta}}{\beta}r + \left(\frac{\bar{\theta}}{\beta}\right)^2\right] \\ &= -3\beta^2\left(r - \frac{\bar{\theta}}{\beta}\right)^2 \\ &< 0. \end{aligned}$$

So $U'_G(r) < 0$ in case H.

Example 1 (ctd.)

As remarked at the end of Section 3, BC (p. 8) show that if $\alpha = 0$ and $\bar{\theta} > 3\beta$, we have $R_G(r) > R_I(r)$ for $r < \bar{\theta}/(3\beta)$. Using the results derived in the proof of Proposition 3 below, we find that $R_G(r)$ attains its global maximum between $r = \bar{\theta}/(3\beta)$ and $r = \bar{\theta}/(2\beta)$. So the situation in which IL is the equilibrium mode of finance, even though $r_G < r_I$ arises if, and only if, $U_I > U_G$ for $r < \bar{\theta}/(2\beta)$. The figure below applies to the extended Example 1.



How likely is IL in equilibrium despite $r_G < r_I$?

The table below summarizes the mean proportion of instances of IL despite $r_G < r_I$ in the 30 cases.

Case no.	\bar \theta_max	\beta_max	# subcases with trade	# subcases with no GL	# subcases with not r_G<r_I	# subcases with r_G<r_I	average proportion of r_G<r_I subcases with IL	
1	4	1,5	125	82	10	33	82,5%	0,4%
2	5	1,5	340	165	14	161	67,2%	1,6%
3	6	1,5	556	186	11	359	56,6%	2,9%
4	7	1,5	697	152	10	535	46,6%	3,6%
5	8	1,5	793	128	7	658	38,6%	3,7%
6	9	1,5	864	111	9	744	33,2%	3,6%
7	4	2	74	47	5	22	81,3%	0,3%
8	5	2	187	85	10	92	65,4%	0,9%
9	6	2	323	120	5	198	55,8%	1,6%
10	7	2	456	137	7	312	49,1%	2,2%
11	8	2	587	149	4	434	45,2%	2,8%
12	9	2	687	131	7	549	41,3%	3,3%
13	4	2,5	57	35	5	17	77,7%	0,2%
14	5	2,5	137	62	6	69	63,4%	0,6%
15	6	2,5	227	80	3	144	54,8%	1,1%
16	7	2,5	318	93	4	221	48,2%	1,5%
17	8	2,5	413	110	2	301	43,9%	1,9%
18	9	2,5	513	126	2	385	41,4%	2,3%
19	4	3	51	31	4	16	76,4%	0,2%
20	5	3	112	47	7	58	61,7%	0,5%
21	6	3	183	63	5	115	52,9%	0,9%
22	7	3	252	72	4	176	47,4%	1,2%
23	8	3	323	82	3	238	43,6%	1,5%
24	9	3	398	92	3	303	41,2%	1,8%
25	4	3,5	45	26	3	16	76,4%	0,2%
26	5	3,5	96	39	4	53	61,2%	0,5%
27	6	3,5	154	51	3	100	52,3%	0,8%
28	7	3,5	212	61	3	148	46,3%	1,0%
29	8	3,5	267	66	2	199	42,8%	1,2%
30	9	3,5	328	74	5	249	40,1%	1,4%
						6905	54,5% unweighted average	45,7% weighted average

Proof that $R_G(r) > R_I(r)$ for $\underline{\theta}/\beta < r < \bar{\theta}/\beta$ when $\alpha = 1$:

Case L

Let $R_{I1}(r)$ and $R_{GL1}(r)$ denote the functions on the the right-hand sides of (6) and (8) for $\alpha = 1$, respectively:

$$R_{I1}(r) = \frac{-\beta r^2 + 2\bar{\theta} r - \frac{\underline{\theta}^2}{\beta}}{2(\bar{\theta} - \underline{\theta})}$$

$$R_{GL1}(r) = \frac{-\beta^2 r^3 + 4\beta\underline{\theta} r^2 + 2(\bar{\theta}^2 - 2\underline{\theta}\bar{\theta} - \underline{\theta}^2) r + \frac{\underline{\theta}^3}{\beta}}{2(\bar{\theta} - \underline{\theta})^2}.$$

As shown above,

$$R_{I1}\left(\frac{\underline{\theta}}{\beta}\right) = R_{GL1}\left(\frac{\underline{\theta}}{\beta}\right) = \frac{\underline{\theta}}{\beta}.$$

Differentiating $R_{I1}(r)$ and $R_{GL1}(r)$ gives

$$R'_{I1}(r) = \frac{-2\beta r + 2\bar{\theta}}{2(\bar{\theta} - \underline{\theta})}$$

$$R'_{GL1}(r) = \frac{-3\beta^2 r^2 + 8\beta\underline{\theta} r + 2(\bar{\theta}^2 - 2\underline{\theta}\bar{\theta} - \underline{\theta}^2)}{2(\bar{\theta} - \underline{\theta})^2}.$$

Evaluating the derivatives at $r = \underline{\theta}/\beta$ yields

$$\begin{aligned} R'_{I1}\left(\frac{\underline{\theta}}{\beta}\right) &= \frac{-2\underline{\theta} + 2\bar{\theta}}{2(\bar{\theta} - \underline{\theta})} \\ &= 1 \\ R'_{GL1}\left(\frac{\underline{\theta}}{\beta}\right) &= \frac{-3\underline{\theta}^2 + 8\underline{\theta}^2 + 2\bar{\theta}^2 - 4\underline{\theta}\bar{\theta} - 2\underline{\theta}^2}{2(\bar{\theta} - \underline{\theta})^2} \\ &= \frac{3\underline{\theta}^2 + 2\bar{\theta}^2 - 4\underline{\theta}\bar{\theta}}{2(\bar{\theta} - \underline{\theta})^2}. \end{aligned}$$

It follows that $R'_{GL1}(\underline{\theta}/\beta) > R'_{I1}(\underline{\theta}/\beta)$:

$$\begin{aligned} R'_{GL1}\left(\frac{\underline{\theta}}{\beta}\right) &> R'_{I1}\left(\frac{\underline{\theta}}{\beta}\right) \\ \frac{3\underline{\theta}^2 + 2\bar{\theta}^2 - 4\underline{\theta}\bar{\theta}}{2(\bar{\theta} - \underline{\theta})^2} &> 1 \\ 3\underline{\theta}^2 + 2\bar{\theta}^2 - 4\underline{\theta}\bar{\theta} &> 2(\bar{\theta} - \underline{\theta})^2 \\ &= 2\bar{\theta}^2 - 4\underline{\theta}\bar{\theta} + 2\underline{\theta}^2 \\ \underline{\theta}^2 &> 0. \end{aligned}$$

That is, $R_{GL1}(r)$ intersects $R_{I1}(r)$ from below at $r = \underline{\theta}/\beta$. Since

$$R_{GL1}(0) = \frac{\underline{\theta}^3}{2\beta(\bar{\theta} - \underline{\theta})^2} > 0 > -\frac{\underline{\theta}^2}{2\beta(\bar{\theta} - \underline{\theta})} = R_{I1}(0),$$

there is an intersection of $R_{GL1}(r)$ and $R_{I1}(r)$ at some r between 0 and $\underline{\theta}/\beta$. Furthermore, we have

$$\begin{aligned} R_{I1}\left(\frac{\bar{\theta}}{\beta}\right) &= \frac{1}{\beta}\frac{\bar{\theta} + \underline{\theta}}{2} \\ R_{GL1}\left(\frac{\bar{\theta}}{\beta}\right) &= \frac{-\frac{\bar{\theta}^3}{\beta} + 4\frac{\underline{\theta}\bar{\theta}^2}{\beta} + 2\frac{\bar{\theta}^3}{\beta} - 4\frac{\underline{\theta}\bar{\theta}^2}{\beta} - 2\frac{\underline{\theta}^2\bar{\theta}}{\beta} + \frac{\underline{\theta}^3}{\beta}}{2(\bar{\theta} - \underline{\theta})^2} \\ &= \frac{1}{\beta}\frac{\bar{\theta}^3 + \underline{\theta}^3 - 2\underline{\theta}^2\bar{\theta}}{2(\bar{\theta} - \underline{\theta})^2} \\ &= \frac{1}{\beta}\frac{\bar{\theta}^2 - \underline{\theta}^2 + \underline{\theta}\bar{\theta}}{2(\bar{\theta} - \underline{\theta})}. \end{aligned}$$

So $R_{GL1}(\bar{\theta}/\beta) > R_{I1}(\bar{\theta}/\beta)$:

$$\begin{aligned} R_{GL1}\left(\frac{\bar{\theta}}{\beta}\right) &> R_{I1}\left(\frac{\bar{\theta}}{\beta}\right) \\ \frac{1}{\beta}\frac{\bar{\theta}^2 - \underline{\theta}^2 + \underline{\theta}\bar{\theta}}{2(\bar{\theta} - \underline{\theta})} &> \frac{1}{\beta}\frac{\bar{\theta}^2 - \underline{\theta}^2}{2(\bar{\theta} - \underline{\theta})} \\ \bar{\theta}^2 - \underline{\theta}^2 + \underline{\theta}\bar{\theta} &> \bar{\theta}^2 - \underline{\theta}^2 \\ \underline{\theta}\bar{\theta} &> 0. \end{aligned}$$

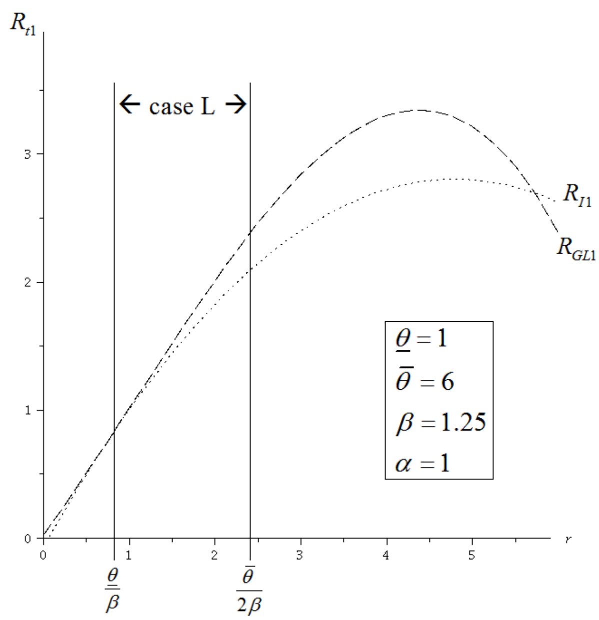
Since $R_{GL1}(r) < R_{I1}(r)$ as r grows large, there is an intersection of $R_{GL1}(r)$ and $R_{I1}(r)$ at some $r > \bar{\theta}/\beta$. So we have identified three points of intersection of $R_{I1}(r)$ and $R_{GL1}(r)$. Since $R_{I1}(r)$ and $R_{GL1}(r)$ are second-order and third-order polynomials, respectively, there cannot be any further intersections, so that

$$R_{GL1}(r) > R_{I1}(r), \quad \frac{\underline{\theta}}{\beta} < r \leq \frac{\bar{\theta}}{\beta}.$$

From the definition of case L (i.e., $\underline{\theta}/\beta \leq r \leq \bar{\theta}/(2\beta)$), it follows that except at the lower boundary $r = \underline{\theta}/\beta$ we have

$$R_G(r) > R_I(r), \quad \text{case L}.$$

An example is illustrated in the figure below.



Case H

Let $R_{GH1}(r)$ denote the function on the the right-hand side of (10) for $\alpha = 1$:

$$R_{GH1}(r) = \frac{\beta^2 r^3 - \beta\bar{\theta}3r^2 + \bar{\theta}^2 3r - \frac{1}{\beta}(\underline{\theta}^2\bar{\theta} + \underline{\theta}\bar{\theta}^2 - \underline{\theta}^3)}{2(\bar{\theta} - \underline{\theta})^2}.$$

Evaluating $R_{GH1}(r)$ at $r = \underline{\theta}/\beta$ yields

$$\begin{aligned}
R_{GH1}\left(\frac{\underline{\theta}}{\beta}\right) &= \frac{\beta^2\left(\frac{\underline{\theta}}{\beta}\right)^3 - 3\beta\bar{\theta}\left(\frac{\underline{\theta}}{\beta}\right)^2 + 3\bar{\theta}^2\frac{\underline{\theta}}{\beta} - \frac{1}{\beta}(\underline{\theta}^2\bar{\theta} + \underline{\theta}\bar{\theta}^2 - \underline{\theta}^3)}{2(\bar{\theta}-\underline{\theta})^2} \\
&= \frac{\underline{\theta}}{\beta}\frac{\underline{\theta}^2 - 3\underline{\theta}\bar{\theta} + 3\bar{\theta}^2 - \underline{\theta}\bar{\theta} - \bar{\theta}^2 + \underline{\theta}^2}{2(\bar{\theta}-\underline{\theta})^2} \\
&= \frac{\underline{\theta}}{\beta}\frac{2\underline{\theta}^2 + 2\bar{\theta}^2 - 4\underline{\theta}\bar{\theta}}{2(\bar{\theta}-\underline{\theta})^2} \\
&= \frac{\underline{\theta}}{\beta} \\
&= R_{I1}\left(\frac{\underline{\theta}}{\beta}\right).
\end{aligned}$$

We have seen in the main text that

$$R_{GH1}\left(\frac{\bar{\theta}}{\beta}\right) = R_{I1}\left(\frac{\bar{\theta}}{\beta}\right) = \frac{1}{\beta}\frac{\underline{\theta}+\bar{\theta}}{2}.$$

Differentiating $R_{GH1}(r)$ gives

$$\begin{aligned}
R'_{GH1}(r) &= \frac{3\beta^2 r^2 - 6\beta\bar{\theta}r + 3\bar{\theta}^2}{2(\bar{\theta}-\underline{\theta})^2} \\
&= \frac{3\beta^2\left(r - \frac{\bar{\theta}}{\beta}\right)^2}{2(\bar{\theta}-\underline{\theta})^2}.
\end{aligned}$$

Evaluating $R'_{I1}(r)$ and $R'_{GH1}(r)$ at $r = \bar{\theta}/\beta$ gives

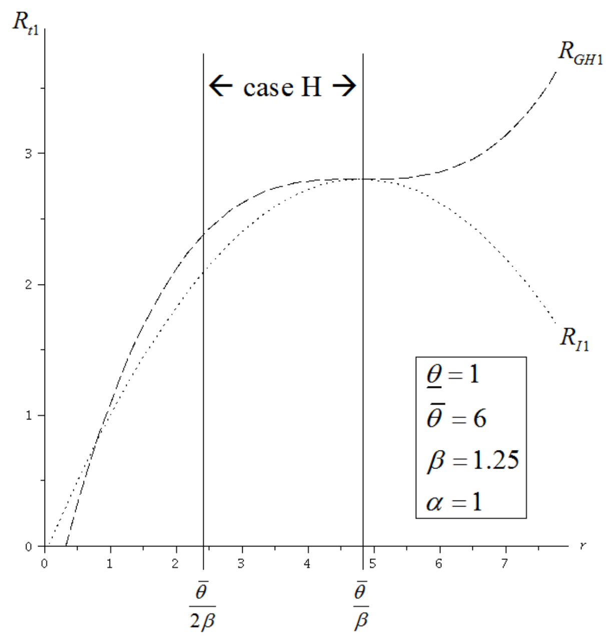
$$R'_{I1}\left(\frac{\bar{\theta}}{\beta}\right) = R'_{GH1}\left(\frac{\bar{\theta}}{\beta}\right) = 0.$$

So $R_{I1}(r)$ and $R_{GH1}(r)$ have one intersection at $r = \underline{\theta}/\beta$ and a "double intersection" at $r = \bar{\theta}/\beta$. Given that $R_{I1}(r)$ and $R_{GH1}(r)$ are second-order and third-order polynomials, respectively, there are no further intersections. It follows that

$$R_{GH1}(r) > R_{I1}(r), \quad \frac{\underline{\theta}}{\beta} < r < \frac{\bar{\theta}}{\beta}.$$

So except at the $r = \bar{\theta}/\beta$ we have

$$R_G(r) > R_I(r), \quad \text{case H.}$$



Proof of Proposition 3:

Define the right-hand side of (6) for $\alpha = 0$ as $R_{I0}(r)$:

$$
\begin{aligned}
R_{I0}(r) &= \frac{-2\beta r^2 + 2\bar{\theta} r}{2(\bar{\theta} - \underline{\theta})} \\
R'_{I0}(r) &= \frac{-4\beta r + 2\bar{\theta}}{2(\bar{\theta} - \underline{\theta})}.
\end{aligned}
$$

$r = \bar{\theta}/(2\beta)$ maximizes $R_{I0}(r)$. The maximum value is

$$
\begin{aligned}
R_{I0}\left(\frac{\bar{\theta}}{2\beta}\right) &= \frac{-2\beta\left(\frac{\bar{\theta}}{2\beta}\right)^2 + 2\bar{\theta}\left(\frac{\bar{\theta}}{2\beta}\right)}{2(\bar{\theta} - \underline{\theta})} \\
&= \frac{\bar{\theta}^2}{4\beta(\bar{\theta} - \underline{\theta})}.
\end{aligned}
$$

To prove Proposition 3, we have to show that $R_G(r)$ cannot exceed this value.

Case L

Let $R_{GL0}(r)$ denote the right-hand side of (8) for $\alpha = 0$:

$$
R_{GL0}(r) = \frac{-6\beta^2 r^3 + 8\beta\underline{\theta} r^2 + 2(\bar{\theta}^2 - 2\underline{\theta}\bar{\theta})r}{2(\bar{\theta} - \underline{\theta})^2}
$$

We have to show that

$$
R_{GL0}(r) < \frac{\bar{\theta}^2}{4\beta(\bar{\theta} - \underline{\theta})}, \quad \text{case L.}
$$

Notice that

$$R_{I0}(0) = R_{GL0}(0) = 0.$$

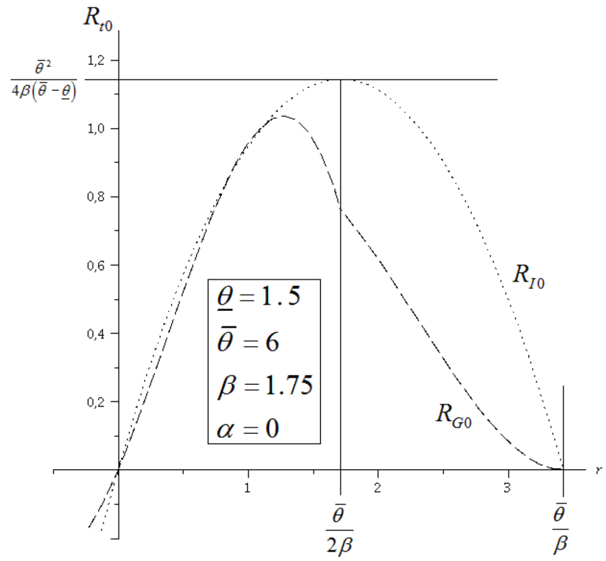
We have $R_{GL0}(r) < R_{I0}(r)$ iff

$$\begin{aligned}
\frac{-6\beta^2 r^3 + 8\beta\underline{\theta} r^2 + 2(\bar{\theta}^2 - 2\underline{\theta}\bar{\theta})r}{2(\bar{\theta} - \underline{\theta})^2} &< \frac{-2\beta r^2 + 2\bar{\theta} r}{2(\bar{\theta} - \underline{\theta})} \\
-6\beta^2 r^2 + 8\beta\underline{\theta} r + 2(\bar{\theta}^2 - 2\underline{\theta}\bar{\theta}) &< (-2\beta r + 2\bar{\theta})(\bar{\theta} - \underline{\theta}) \\
&= -2\beta(\bar{\theta} - \underline{\theta})r + 2(\bar{\theta}^2 - \underline{\theta}\bar{\theta}) \\
-6\beta^2 r^2 + 8\beta\underline{\theta} r - 2\underline{\theta}\bar{\theta} &< -2\beta(\bar{\theta} - \underline{\theta})r \\
6\beta^2 r^2 - 2\beta(\bar{\theta} + 3\underline{\theta})r + 2\underline{\theta}\bar{\theta} &> 0 \\
r^2 - \frac{\bar{\theta} + 3\underline{\theta}}{3\beta} r + \frac{\underline{\theta}\bar{\theta}}{3\beta^2} &> 0.
\end{aligned}$$

The roots of the quadratic equation on the left-hand side are:

$$\begin{aligned}
r_{1/2} &= \frac{\bar{\theta} + 3\underline{\theta}}{6\beta} \pm \sqrt{\frac{1}{4}\left(\frac{\bar{\theta} + 3\underline{\theta}}{3\beta}\right)^2 - \frac{\underline{\theta}\bar{\theta}}{3\beta^2}} \\
&= \frac{\bar{\theta} + 3\underline{\theta}}{6\beta} \pm \frac{1}{6\beta}\sqrt{\bar{\theta}^2 - 6\underline{\theta}\bar{\theta} + 9\underline{\theta}^2} \\
&= \frac{\bar{\theta} + 3\underline{\theta}}{6\beta} \pm \frac{1}{6\beta}\sqrt{(\bar{\theta} - 3\underline{\theta})^2} \\
&= \frac{(\bar{\theta} + 3\underline{\theta}) \pm (\bar{\theta} - 3\underline{\theta})}{6\beta} \\
&= \left\{\frac{\underline{\theta}}{\beta}, \frac{\bar{\theta}}{3\beta}\right\} \\
&< \frac{\bar{\theta}}{2\beta}.
\end{aligned}$$

So the three intersections of $R_{GL0}(r)$ and $R_{I0}(r)$ occur at $r = 0$, $r = r_1 = \underline{\theta}/\beta$, and $r = r_2 = \bar{\theta}/(3\beta)$.



Suppose first that $\bar{\theta} \leq 3\underline{\theta}$, so that

$$r_2 = \frac{\bar{\theta}}{3\beta} < \frac{\underline{\theta}}{\beta} = r_1.$$

Then

$$R_{GL0}(r) < R_{I0}(r), \quad r > \frac{\underline{\theta}}{\beta}.$$

Next, consider the case $\bar{\theta} > 3\underline{\theta}$, in which $r_2 = \bar{\theta}/(3\beta) > \underline{\theta}/\beta = r_1$. Differentiating $R_{GL0}(r)$ gives

$$R'_{GL0}(r) = \frac{-18\beta^2 r^2 + 16\beta\underline{\theta}r + 2(\bar{\theta}^2 - 2\underline{\theta}\bar{\theta})}{2(\bar{\theta}-\underline{\theta})^2}.$$

So

$$\begin{aligned}
R'_{GL0}(0) &= \frac{\bar{\theta}(\bar{\theta}-2\underline{\theta}}{(\bar{\theta}-\underline{\theta})^2} \\
&> 0 \\
R'_{GL0}\left(\frac{\bar{\theta}}{3\beta}\right) &= \frac{-18\beta^2\left(\frac{\bar{\theta}}{3\beta}\right)^2 + 16\beta\underline{\theta}\frac{\bar{\theta}}{3\beta} + 2(\bar{\theta}^2 - 2\underline{\theta}\bar{\theta})}{2(\bar{\theta}-\underline{\theta})^2} \\
&= \frac{-2\bar{\theta}^2 + \frac{16}{3}\bar{\theta}\underline{\theta} + 2\bar{\theta}^2 - 4\bar{\theta}\underline{\theta}}{2(\bar{\theta}-\underline{\theta})^2} \\
&= \frac{2\bar{\theta}\underline{\theta}}{3(\bar{\theta}-\underline{\theta})^2} \\
&> 0.
\end{aligned}$$

The fact that $R_{GL0}(r)$ is upward-sloping at $r=0$ and at $r = \bar{\theta}/(3\beta)$ implies that it is upward-sloping in between. It follows that

$$R_{GL0}(r) < R_{GL0}\left(\frac{\bar{\theta}}{3\beta}\right) = R_I\left(\frac{\bar{\theta}}{3\beta}\right), \quad r < \frac{\bar{\theta}}{3\beta}$$

and

$$R_{GL0}(r) < R_{I0}(r), \quad r > \frac{\bar{\theta}}{3\beta}.$$

This completes the proof that

$$R_{GL0}(r) < \frac{\bar{\theta}^2}{4\beta(\bar{\theta} - \underline{\theta})}, \quad \text{case L.}$$

Case H

Let $R_{GH0}(r)$ denote the right-hand side of (10) for $\alpha = 0$:

$$\begin{aligned} R_{GH0}(r) &= \frac{2\beta^2 r^3 - 4\beta\bar{\theta}r^2 + 2\bar{\theta}^2 r}{2(\bar{\theta} - \underline{\theta})^2} \\ R'_{GH0}(r) &= \frac{6\beta^2 r^2 - 8\beta\bar{\theta}r + 2\bar{\theta}^2}{2(\bar{\theta} - \underline{\theta})^2}. \end{aligned}$$

We have $R'_{GH0}(r) = 0$ iff

$$\begin{aligned} 0 &= 6\beta^2 r^2 - 8\beta\bar{\theta}r + 2\bar{\theta}^2 \\ 0 &= r^2 - \frac{4}{3}\frac{\bar{\theta}}{\beta}r + \frac{1}{3}\left(\frac{\bar{\theta}}{\beta}\right)^2 \\ r_{1/2} &= \frac{2}{3}\frac{\bar{\theta}}{\beta} \pm \sqrt{\frac{4}{9}\left(\frac{\bar{\theta}}{\beta}\right)^2 - \frac{1}{3}\left(\frac{\bar{\theta}}{\beta}\right)^2} \\ &= \frac{2}{3}\frac{\bar{\theta}}{\beta} \pm \frac{1}{3}\frac{\bar{\theta}}{\beta} \\ &= \left\{\frac{\bar{\theta}}{3\beta}, \frac{\bar{\theta}}{\beta}\right\}. \end{aligned}$$

$R_{GH0}(r)$ is downward-sloping in the interval $(\bar{\theta}/(2\beta), \bar{\theta}/\beta)$, so

$$R_{GH0}(r) \leq R_{GH0}\left(\frac{\bar{\theta}}{2\beta}\right), \quad \text{case H.}$$

From coninuity of $R_G(r)$,

$$R_{GH0}\left(\frac{\bar{\theta}}{2\beta}\right) = R_{GL0}\left(\frac{\bar{\theta}}{2\beta}\right).$$

Using

$$R_{GL0}\left(\frac{\bar{\theta}}{2\beta}\right) < \frac{\bar{\theta}^2}{4\beta(\bar{\theta} - \underline{\theta})},$$

it follows that

$$R_{GL0}(r) < \frac{\bar{\theta}^2}{4\beta(\bar{\theta} - \underline{\theta})}, \quad \text{case H.}$$

With regard to example 1, we get

$$R'_{GL0}\left(\frac{\bar{\theta}}{2\beta}\right) = -\frac{\frac{5}{2}\left(\bar{\theta} - \frac{8}{5}\underline{\theta}\right)}{2(\bar{\theta} - \underline{\theta})^2} < 0.$$

So $R_{GL0}(r)$ attains a maximum between $r = \bar{\theta}/(3\beta)$ and $r = \bar{\theta}/(2\beta)$. The maximizing r-value can be calculated explicitly:

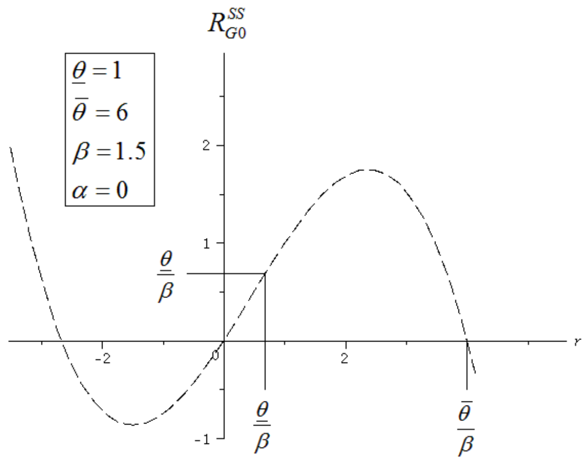
$$\begin{aligned}
0 &= R'_{GL0}(r) \\
0 &= -18\beta^2 r^2 + 16\beta\underline{\theta}r + 2(\bar{\theta}^2 - 2\underline{\theta}\bar{\theta}) \\
0 &= r^2 - \frac{8}{9}\frac{\underline{\theta}}{\beta}r - \frac{\bar{\theta}^2 - 2\underline{\theta}\bar{\theta}}{9\beta^2} \\
r_{1/2} &= \frac{4}{9}\frac{\underline{\theta}}{\beta} \pm \left[\frac{1}{4}\left(\frac{8}{9}\frac{\underline{\theta}}{\beta}\right)^2 + \frac{\bar{\theta}^2 - 2\underline{\theta}\bar{\theta}}{9\beta^2}\right]^{\frac{1}{2}} \\
&= \frac{4}{9}\frac{\underline{\theta}}{\beta}\left[1 \pm \left(1 + \frac{9}{16}\frac{\bar{\theta}^2 - 2\underline{\theta}\bar{\theta}}{\underline{\theta}^2}\right)^{\frac{1}{2}}\right]
\end{aligned}$$

The smaller root is negative, the larger positive: $r_1 < 0 < r_2$.

Proof that $R_G(r)$ is hump-shaped in the model with social sanctions

$$\begin{aligned}
R_G\left(\frac{\underline{\theta}}{\beta}\right) &= \Pi_G\left(\frac{\underline{\theta}}{\beta}\right)\frac{\underline{\theta}}{\beta} \\
&= \frac{\underline{\theta}}{\beta} \\
R_G\left(\frac{\bar{\theta}}{\beta}\right) &= \Pi_G\left(\frac{\bar{\theta}}{\beta}\right)\frac{\bar{\theta}}{\beta} \\
&= 0 \\
R_G(r) &= \Pi_G(r)r \\
&= \frac{-\beta^2 r^3 + 2\beta\underline{\theta}r^2 + (\bar{\theta}^2 - 2\underline{\theta}\bar{\theta})r}{(\bar{\theta} - \underline{\theta})^2} \\
R'_G(r) &= \frac{-3\beta^2 r^2 + 4\beta\underline{\theta}r + \bar{\theta}^2 - 2\underline{\theta}\bar{\theta}}{(\bar{\theta} - \underline{\theta})^2} \\
R'_G\left(\frac{\underline{\theta}}{\beta}\right) &= \frac{-3\beta^2\left(\frac{\underline{\theta}}{\beta}\right)^2 + 4\beta\underline{\theta}\frac{\underline{\theta}}{\beta} + \bar{\theta}^2 - 2\underline{\theta}\bar{\theta}}{(\bar{\theta} - \underline{\theta})^2} \\
&= \frac{-3\underline{\theta}^2 + 4\underline{\theta}^2 + \bar{\theta}^2 - 2\underline{\theta}\bar{\theta}}{(\bar{\theta} - \underline{\theta})^2} \\
&= \frac{(\bar{\theta} - \underline{\theta})^2}{(\bar{\theta} - \underline{\theta})^2} \\
&= 1 \\
&> 0.
\end{aligned}$$

The fact that $R_G(r)$ has no root in the interval $(\underline{\theta}/\beta, \bar{\theta}/\beta)$ and that $R_G(r)$ is positive and upward-sloping at $r = \underline{\theta}/\beta$ and equals zero for $r = \bar{\theta}/\beta$ implies that it is positive and takes on a unique local maximum in the interval.



Proof of Proposition 6

Expected borrower utility with lending type t is

$$\begin{aligned} U_t(r) &= \Pi_t(r)E(\theta - r|\,\boldsymbol{\theta}_t \in S_t) + [1-\Pi_t(r)]E\left(\theta - \frac{\theta}{\beta}\right|\,\boldsymbol{\theta}_t \in D_t\Big) \\ &= E(\theta) - \Pi_t(r)r - \frac{1}{\beta}[1-\Pi_t(r)]E(\theta|\,\boldsymbol{\theta}_t \in D_t). \end{aligned}$$

Define the last term on the right-hand side as the deadweight loss with lending type t:

$$L_t(r) \equiv \frac{1}{\beta}[1-\Pi_t(r)]E(\theta|\,\boldsymbol{\theta}_t \in D_t), \quad t \in \{I, G\}.$$

Then

$$U_t(r) = E(\theta) - \Pi_t(r)r - L_t(r).$$

For GL, the conditional expectation of θ given default is

$$\begin{aligned} E(\theta|\,\boldsymbol{\theta}_G \in D_G) &= \frac{1}{1-\Pi_G(r)} \int_{\underline{\theta}}^{\beta r} \frac{1}{\bar{\theta}-\underline{\theta}} \int_{\underline{\theta}}^{\beta r} \frac{\theta}{\bar{\theta}-\underline{\theta}} d\theta\, d\theta' \\ &= \frac{1}{1-\Pi_G(r)} \frac{\int_{\underline{\theta}}^{\beta r} d\theta' \int_{\underline{\theta}}^{\beta r} \theta d\theta}{(\bar{\theta}-\underline{\theta})^2} \\ &= \frac{1}{1-\Pi_G(r)} \frac{(\beta r - \underline{\theta})(\beta^2 r^2 - \underline{\theta}^2)}{2(\bar{\theta}-\underline{\theta})^2} \\ &= \frac{1}{1-\Pi_G(r)} \frac{\beta^3 r^3 - \beta^2 \underline{\theta} r^2 - \beta \underline{\theta}^2 r + \underline{\theta}^3}{2(\bar{\theta}-\underline{\theta})^2}. \end{aligned}$$

(It is easy to check that this conditional expectation can be written as $(\beta r + \underline{\theta})/2$.)

Let $\max_r R_G(r) \geq \rho$, so that r_G is well defined. By definition, $\Pi_G(r_G)r_G = \rho$. Suppose there is a different interest rate r ($> r_G$) which borrowers prefer and which yields positive profits for banks: $\Pi_G(r)r > \rho$. From

$$U_t(r) = E(\theta) - \Pi_t(r)r - L_t(r),$$

this implies

$$L_t(r) < L_t(r_G)$$

for $t = G$. Since $r > r_G$, we have a contradiction if

$$\begin{aligned}
L'_G(r) &> 0 \\
\frac{d}{dr}\left\{\frac{1}{\beta}[1 - \Pi_G(r)]E(\theta|\,\boldsymbol{\theta}_G \in D_G)\right\} &> 0 \\
\frac{d}{dr}\left[\frac{\beta^3 r^3 - \beta^2\underline{\theta}r^2 - \beta\underline{\theta}^2 r + \underline{\theta}^3}{2(\bar{\theta} - \underline{\theta})^2}\right] &> 0 \\
\frac{d}{dr}[\beta^3 r^3 - \beta^2\underline{\theta}r^2 - \beta\underline{\theta}^2 r + \underline{\theta}^3] &> 0 \\
3\beta^3 r^2 - 2\beta^2\underline{\theta}r - \beta\underline{\theta}^2 &> 0 \\
r^2 - \frac{2}{3}\frac{\underline{\theta}}{\beta}r - \frac{1}{3}\left(\frac{\underline{\theta}}{\beta}\right)^2 &> 0.
\end{aligned}$$

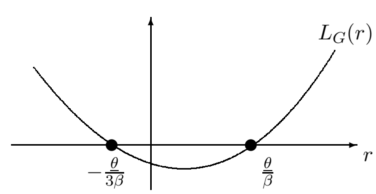
The roots of the quadratic equation on the left-hand side are

$$\begin{aligned}
r_{1/2} &= \frac{1}{3}\frac{\underline{\theta}}{\beta} \pm \left[\frac{1}{9}\left(\frac{\underline{\theta}}{\beta}\right)^2 + \frac{1}{3}\left(\frac{\underline{\theta}}{\beta}\right)^2\right]^{\frac{1}{2}} \\
&= \frac{1}{3}\frac{\underline{\theta}}{\beta} \pm \frac{2}{3}\frac{\underline{\theta}}{\beta} \\
&= \left\{-\frac{1}{3}\frac{\underline{\theta}}{\beta}, \frac{\underline{\theta}}{\beta}\right\}.
\end{aligned}$$

Evidently, we have

$$r^2 - \frac{2}{3}\frac{\underline{\theta}}{\beta}r - \frac{1}{3}\left(\frac{\underline{\theta}}{\beta}\right)^2 > 0, \quad r > \frac{\underline{\theta}}{\beta}.$$

This proves that there is no profitable GL contract.



As to IL contracts, note that

$$\begin{aligned}
E(\theta|\boldsymbol{\theta}_I \in D_I) &= \frac{1}{1 - \Pi_I(r)}\int_{\underline{\theta}}^{\beta r}\frac{\theta}{\bar{\theta} - \underline{\theta}}d\theta \\
&= \frac{1}{1 - \Pi_I(r)}\frac{\beta^2 r^2 - \underline{\theta}^2}{2(\bar{\theta} - \underline{\theta})} \\
E(\theta|\boldsymbol{\theta}_I \in D_I)[1 - \Pi_I(r)] &= \frac{\beta^2 r^2 - \underline{\theta}^2}{2(\bar{\theta} - \underline{\theta})} \\
L_I(r) &= \frac{1}{\beta}\frac{\beta^2 r^2 - \underline{\theta}^2}{2(\bar{\theta} - \underline{\theta})}.
\end{aligned}$$

$L_I'(r) > 0$ for all $r > 0$. A comparison with

$$L_G(r) = \frac{1}{\beta}\frac{(\beta r - \underline{\theta})(\beta^2 r^2 - \underline{\theta}^2)}{2(\bar{\theta} - \underline{\theta})^2}$$

shows that for all r,

$$L_G(r) < L_I(r).$$

Suppose r_I exists. The fact that $R_G(r) > R_I(r)$ for all r implies $r_I > r_G$. From $L_G'(r) > 0$ and $L_G(r) < L_I(r)$, it follows that

$$L_G(r_G) < L_G(r_I) < L_I(r_I).$$

Using $R_G(r_G) = R_I(r_I) = \rho$, we have

$$\begin{aligned}
L_G(r_G) &< L_I(r_I) \\
-L_G(r_G) &> -L_I(r_I) \\
-\rho - L_G(r_G) &> -\rho - L_I(r_I) \\
-R_G(r_G) - L_G(r_G) &> -R_I(r_I) - L_I(r_I) \\
E(\theta) - R_G(r_G) - L_G(r_G) &> E(\theta) - R_I(r_I) - L_I(r_I) \\
U_G(r_G) &> U_I(r_I).
\end{aligned}$$

That is, borrowers prefer GL at r_G to IL at r_I. As r_I is the lowest IL loan rate that breaks even and $L_I'(r) > 0$, it follows that

$$U_G(r_G) > U_I(r) = E(\theta) - R_I(r) - L_I(r)$$

for all r whenever $R_I(r) > \rho$. So it is not possible to enter the market with IL at a loan rate above r_I either.