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Nicolas Pinkwart

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Nicolas Pinkwart

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Friedrich-Alexander-Universität
Erlangen-Nürnberg

Institut für Wirtschaftswissenschaft
(Institute of Economics)
Kochstraße 4 (17), 91056 Erlangen
Telefon: +49 9131 85-22381
Telefax: +49 9131 85-22060
e-Mail: nicolas.pinkwart@wiwi.phil.uni-erlangen.de
Internet: www.economics.phil.uni-erlangen.de

Abstract

This paper offers several contributions to actual research and discussion on monetary policy. It clarifies the relationship between uncertainty of inflation persistence and optimal monetary policy and discusses the consequences of the recent Blanchard proposal to implement a higher inflation target in the light of parameter uncertainty. Furthermore, it provides insights of general interest on the methodological level by analyzing the interrelations between normalization of variables and their independence properties and by extending standard solution methods of dynamic programming problems to non-orthogonal parameter uncertainty.

JEL: E52, E58, C61.

Keywords: Inflation Persistence, Parameter Uncertainty, Inflation Target, Dynamic Programming.

Summary

While the traditional view on parameter uncertainty comes to the conclusion that monetary policy should be more cautious than under certainty equivalence, one important exception is the case of uncertainty about the persistence of inflation, where optimal policy is found to be more aggressive. However, this result seems to be unclear in the case of strict inflation targeting. The findings by Craine (1979) imply greater aggression, whereas Söderström (2002) finds certainty equivalence. This paper reconciles the discrepancy in the literature by thorough examination of the interdependencies between normalization and imposing independence assumptions. Since both procedures are commonly used in economic analysis, the findings are of general interest from a methodological point of view. By extending the solution method of the standard linear regulator problem to multiplicative uncertainty in conjunction with non-orthogonal residuals, it is shown that Söderström's result stems from a certain combination of normalizing and imposing the independence assumption. In general, optimal monetary policy under uncertainty about the persistence of inflation is found to be not certainty equivalent, even in a strict inflation targeting framework. In fact, the neutral stance of monetary policy is affected by uncertainty and differs from its certainty equivalent value. The direction of this deviation depends on the covariance between inflation persistence and the inflation shock. This finding is not restricted to the strict inflation targeting case and therefore has wider implications. Some of these are highlighted by an application to the recent proposal by Blanchard et al. (2010) to increase the inflation target from 2 to 4 percent. It is demonstrated that this target shift in combination with uncertainty about inflation persistence leads to a higher neutral policy interest rate than under certainty equivalence. Conversely, for any reduction of the inflation target, inflation persistence uncertainty would support a lower neutral rate. Thus, inflation persistence uncertainty has interesting diametrical effects in the case of a change in the inflation target: For any shift toward a tighter regime, it supports the "doves" in the central bank's decision committee, while for any loosening in the inflation target it gives backup to the "hawks".

1 Uncertainty about the Persistence of Inflation

Over the past decade, research on monetary policy under uncertainty has brought important insights. As Bernanke (2008) points out, one of the major results is, that the cautious response to economic shocks suggested by analysis in the tradition of Brainard (1967) is not always appropriate. An important exception to the "Brainard Conservatism Principle" (Blinder, 1998, p.17) arises in the case of uncertainty about the persistence of inflation. While the conservatism conclusion is generally valid for uncertain monetary policy transmission parameters, uncertainty about inflation persistence may make the optimal policy response more aggressive.

This result dates back to Craine (1979), who analyzes parameter uncertainty in an univariate model context, and was confirmed more recently in modeling frameworks of higher complexity by Söderström (2002), Moessner (2005) and Kimura and Kurozumi (2007). While the latter studies explore the consequences of parameter uncertainty in micro-founded models with forward-looking expectations, Söderström (2002) shows the optimality of a more aggressive policy in the case of uncertainty about inflation persistence in the purely backward-looking model context of Ball (1997) and Svensson (1997, 1999).

However, to reach this result, Söderström needs a strictly positive weight on output gap stabilization in the central banks loss function. Turning from this "flexible inflation targeting" regime toward "strict inflation targeting" in the sense of Svensson (1999), i.e. assuming a zero-weight on output, optimal policy under uncertainty about inflation persistence is certainty equivalent. This finding seems to be at odds with the one obtained by Craine (1979) who considers only one target variable and whose analysis is thus comparable with the strict inflation targeting case. Craine takes output as the target variable which is modeled as being linearly dependent on its first lag and the monetary policy instrument (the money growth rate). In the case of uncertainty about output dynamics, he finds a more aggressive policy to be optimal. This manifests itself in lower average money growth and thus tighter monetary policy than under certainty equivalence.

Empirical research on inflation persistence shows little consensus about its degree, so there is a substantial amount of uncertainty about inflation dynamics.[1] The issue of its implications for optimal monetary policy is therefore of major importance. Hence, the apparent discrepancy in the results motivates the search for an explanation.

This paper offers a solution for this puzzle and shows how the Söderström (2002) framework can be expanded to a more general analysis of the case of strict inflation

[1] See for example the overview on the results of the Eurosystem Inflation Persistence Network by Altissimo et al. (2006).

targeting. As transformations of variables usually alter the covariances between model parameters and residuals, it turns out that the conventional assumption of independence between model parameters and residuals can not be imposed without hesitation, if the model variables are normalized. Söderström normalizes the inflation rate and the nominal interest rate to zero mean variables by subtracting the respective long run average values and assumes that the parameters and residuals of the transformed model are independent. This implies, that there are non-zero covariances in the original non-normalized model. If the ordering is changed, so that one starts with a non-normalized model with independent parameters and residuals, the demeaning of variables will change the covariance between the parameter of inflation persistence and the inflation-shock (in fact, the covariance will increase). As Brainard (1967) noted and more recent research confirmed (see for example Martin, 1999; Gonzalez, 2008) non-zero covariances can alter the effects of parameter uncertainty on optimal policy. Since the procedures of normalizing variables and imposing independence are widely used in economic analysis, these findings are also of general interest on the methodological level.

It is shown that the combination of normalizing and the assumption of independence in Söderström (2002) leads to a special condition for the covariance between the parameter of inflation persistence and the inflation-shock, which implies certainty equivalence for the case of strict inflation targeting, and that any other specification leads to an optimal policy that is not certainty equivalent. The solution of the optimization problem for the case of non-orthogonality between parameters and residuals requires further generalizations of standard dynamic programming techniques than multiplicative uncertainty alone and thus represents the second methodological contribution of this paper.

In general, parameter uncertainty is shown to cause the neutral policy interest rate[2] to deviate from its value under certainty. For the non-normalized model with independence between parameters and residuals, optimal monetary policy shows a higher neutral policy rate and thus the policy maker fights inflation more aggressively. This "hawkish policy" result is hence again in line with Craine (1979). Moreover, this effect of uncertainty about persistence on the neutral policy rate is not restricted to the strict inflation targeting case but can also be found for any flexible inflation targeting regime.

It thus may be questioned, which ordering of normalization and assuming independence is the "right one". In this paper it is argued, that this question has an ambiguous answer. The implication of parameter uncertainty on the neutral rate depends on the value for the covariance. Any independence assumption, whether made before or after normalization, determines a certain covariance. The point is, that this covariance can

[2]The neutral policy interest rate is measured by the policy rate that is set when inflation is at target and the output gap is zero.

change and thus may cause implications of inflation persistence uncertainty as described above.

To illustrate this result, the modeling framework is used to analyze the recent Blanchard et al. (2010) proposition to increase the inflation target from 2 to 4 percent. It is shown that the new target requires a different normalization process which increases the covariance between inflation persistence parameter and inflation shock. This leads to a shift in the neutral rate to a higher value than under certainty equivalence. So taking uncertainty into account will back up the monetary policy "hawks" and the central bank will gain an even higher policy scope than Blanchard et al. (2010) suggest. This conclusion is symmetric, for there are, of course, also situations possible with opposing effects. Any reduction of the inflation target, for example, will lead to a decreasing covariance and hence increasing parameter uncertainty may lead to a lower neutral stance than under certainty equivalence, giving support to the "doves".

The rest of the paper is organized as follows: Section 2 presents the modeling framework, section 3 studies the interrelations between normalization and the independence assumption. In section 4 the implications of inflation persistence uncertainty with reversed ordering of normalization and independence assumption are analyzed, section 5 discusses the results and highlights their implications for the Blanchard proposal, section 6 concludes.

2 The Model

The Söderström (2002) study uses the following model, which is a version of the (Svensson, 1997, 1999) model, consisting of two structural equations:

First, there is a Phillips-curve relationship between inflation $\hat{\pi}_{t+1}$ and the output gap y_t:

$$\hat{\pi}_{t+1} = \alpha_{t+1}\hat{\pi}_t + \beta_{t+1}y_t + \eta^{\pi}_{t+1} \quad (1)$$

Inflation is measured as deviation from long run average inflation (which is assumed to equal the central banks inflation objective π^*), i.e. $\hat{\pi}_{t+1} = \pi_{t+1} - \pi^*$, where π_{t+1} is the inflation rate of period $t+1$. Thus, next period's inflation is positively related to current inflation and to the current output gap. All other influences are captured by the residual η^{π}_{t+1}.

Second, the link between the output gap and the monetary policy instrument, which is assumed to be the short term nominal interest rate, is described by an IS-curve relationship of the form:

$$(2) \quad y_{t+1} = \delta_{t+1} y_t - \gamma_{t+1}(\hat{i}_t - \hat{\pi}_t) + \eta^y_{t+1}$$

The short term nominal interest rate is also measured in deviations from the long run average $\bar{i}$, i.e. $\hat{i}_t = i_t - \bar{i}$, where i_t is the actual nominal rate. The output gap of next period is positively related to the current output gap and negatively related to the current real interest rate $\hat{i}_t - \hat{\pi}_t$. As before, all remaining influences are captured by the residual η^y_{t+1}.

The residuals η^π_{t+1} and η^y_{t+1} are assumed to be i.i.d. shocks with zero mean and variances $\sigma^2_{\eta^\pi}$ and $\sigma^2_{\eta^y}$. All model parameters are assumed to be stochastic, so they can be interpreted as i.i.d. random variables with means $E[\alpha_{t+1}] = \alpha$, $E[\beta_{t+1}] = \beta$, $E[\gamma_{t+1}] = \gamma$, $E[\delta_{t+1}] = \delta$ and variances σ^2_α, σ^2_β, σ^2_γ, σ^2_δ.

The central bank sets its instrument $\hat{i}_t$ to minimize the discounted future loss caused by deviations of inflation and output from their targets that it expects given current available information. So it minimizes

$$(3) \quad E_t[\sum_{j=t}^{\infty} \omega^{j-t}(\hat{\pi}^2_j + \lambda_y \cdot y^2_j)]$$

over all possible paths $\{\hat{i}_j\}^\infty_{j=t}$. $E_t[\cdot]$ denotes the conditional expectation operator and $0 \leq \omega < 1$ is the central banks discount factor. The weight that the central bank puts on output stabilization relative to inflation stabilization is given by $\lambda_y \geq 0$. The loss function has the usual quadratic structure. For we use the normalized variables for inflation and output gap, the central banks optimization problem is a version of the standard linear regulator problem as described for example by Ljungqvist and Sargent (2004) that is modified for multiplicative uncertainty and can be resolved by dynamic programming methods. Since the model has a simple one-lag structure, the resulting optimal interest rate rule is of the Taylor (1993) form. In this framework, Söderström (2002) shows that for $\lambda_y > 0$, uncertainty about the inflation persistence parameter α_{t+1} leads to a more aggressive optimal monetary policy than under certainty. For $\lambda_y = 0$, however, optimal policy is found to be certainty equivalent.

3 Normalization and the Independence Condition

Next, it is demonstrated, how normalization, as it is done by demeaning the variables π_t and i_t in the Söderström (2002) analysis, alters the covariances between model parameters and residuals. Let the Söderström model (1), (2) be given in a non-normalized form (henceforth called the original model):

$$(4) \quad \pi_{t+1} = \alpha_{t+1}\pi_t + \beta_{t+1}y_t + \varepsilon^{\pi}_{t+1}$$

$$(5) \quad y_{t+1} = \delta_{t+1}y_t - \gamma_{t+1}(i_t - \pi_t) + \varepsilon^{y}_{t+1}$$

Here, the original residuals are denoted by ε^{π}_{t+1} and ε^{y}_{t+1} to emphasize the difference to residuals of the normalized model. We do not impose any restrictions on covariances, so these may be given as $\sigma_{\alpha\beta}$, $\sigma_{\alpha\varepsilon^{\pi}}$, $\sigma_{\beta\varepsilon^{\pi}}$, $\sigma_{\gamma\delta}$, $\sigma_{\gamma\varepsilon^{y}}$ and $\sigma_{\delta\varepsilon^{y}}$. Moreover, given are the long run average values of inflation (π^*) and the central bank's instrument ($\bar{i} \geq 0$). To guarantee, that these values together with a long run output gap of zero are indeed the long run average values of the model, they need to equal the steady state values implied by (4) and (5). This means, that the long run values must make the model equations hold in the absence of any stochastic shocks. Hence, the unconditional expectations for the additive shocks are determined by:

$$(6) \quad \varepsilon^{\pi} = (1-\alpha)\pi^*$$

$$(7) \quad \varepsilon^{y} = \gamma(\bar{i} - \pi^*)$$

We now perform the normalization procedure like Söderström, bringing Model (4), (5) into its equivalent form

$$(4') \quad \pi_{t+1} - \pi^* = \alpha_{t+1}(\pi_t - \pi^*) + \beta_{t+1}y_t + \alpha_{t+1}\pi^* - \pi^* + \varepsilon^{\pi}_{t+1}$$

(5') $y_{t+1} = \delta_{t+1} y_t - \gamma_{t+1}(i_t - \bar{i} - (\pi_t - \pi^*)) + \gamma_{t+1}(\pi^* - \bar{i}) + \varepsilon^y_{t+1}$

which can also be written as in (1) and (2) with residuals given by:

(8) $\eta^\pi_{t+1} = (\alpha_{t+1} - 1)\pi^* + \varepsilon^\pi_{t+1}$

(9) $\eta^y_{t+1} = \gamma_{t+1} \cdot (\pi^* - \bar{i}) + \varepsilon^y_{t+1}$

From (6) and (7) it can be seen that the new residuals η^π_{t+1} and η^y_{t+1} have zero means. Their variances are $\sigma^2_{\eta^\pi} = (\pi^*)^2 \cdot \sigma^2_\alpha + \sigma^2_{\varepsilon^\pi}$ and $\sigma^2_{\eta^y} = (\pi^* - \bar{i})^2 \cdot \sigma^2_\gamma + \sigma^2_{\varepsilon^y}$. However, since the residuals now explicitly depend on some of the model parameters, some of the covariances will be affected. While all other covariances remain stable, the covariance between the inflation persistence parameter α_{t+1} and the inflation-shock becomes

(10) $\sigma_{\alpha\eta^\pi} = \pi^* \cdot \sigma^2_\alpha + \sigma_{\alpha\varepsilon^\pi}$

and the covariance between the monetary policy transmission parameter γ_{t+1} and the output gap-shock is now

(11) $\sigma_{\gamma\eta^y} = (\pi^* - \bar{i}) \cdot \sigma^2_\gamma + \sigma_{\gamma\varepsilon^y}$

It follows from (10) that in case of $\pi^* > 0$ and with uncertainty about α_{t+1}, i.e. $\sigma^2_\alpha > 0$, the inequality $\sigma_{\alpha\eta^\pi} > \sigma_{\alpha\varepsilon^\pi}$ holds. Thus, it can be seen that normalization changes the residuals such that the covariance between α_{t+1} and the inflation shock increases in comparison to the original model.[3]

If, for example, we start with the original model (4) and (5) on which we impose the condition that all model parameters and residuals are pairwise independent (this condition will henceforth be called the "independence condition"), then normalization leads to a non-zero covariance between the parameter α_{t+1} and the new inflation shock of $\sigma_{\alpha\eta^\pi} = \pi^* \cdot \sigma^2_\alpha$, which increases with the uncertainty about inflation persistence. As

[3] I focus on $\sigma_{\alpha\eta^\pi}$, because this paper is concerned with uncertainty about inflation persistence. The following arguments can be stated in analogous form for the covariance $\sigma_{\gamma\eta^y}$.

mentioned earlier, a non-zero covariance like this can change the implications of parameter uncertainty on optimal policy.

It is also of importance to notice that (10) implies the following: If the independence condition is assumed for the normalized model (1), (2) as in Söderström (2002), the covariance $\sigma_{\alpha\varepsilon^\pi}$ is equal to $-\pi^* \cdot \sigma_\alpha^2$ which is negative if $\pi^* > 0$ and $\sigma_\alpha^2 > 0$.

It thus follows:

1. The main point of this section is of a general methodological nature: Normalizing of variables needs to be done carefully, for it can have non-trivial consequences. Here, these consequences affect the stochastic properties of the model's parameters and residuals and hence the solution of the central bank's optimization problem. For the implications of uncertainty about the inflation persistence parameter on optimal monetary policy it is important whether the postulation of the independence condition or normalization is done first.

2. If the independence condition is imposed **before** normalization, the normalized model will, in general, show some non-zero covariances (except in some special cases, like $\pi^* = \bar{i} = 0$ in the example above).

3. If the independence condition is imposed **after** normalization, the original non-normalized model will, in general, show some non-zero covariances (except in some special cases, like $\pi^* = \bar{i} = 0$ in the example above).

4. Since non-zero covariances may alter the implications of parameter uncertainty, it is interesting to explore the consequences of imposing the independence condition before normalization for the implications of uncertainty about the dynamics of inflation. A positive covariance between inflation persistence parameter and inflation-shock may suggest a more aggressive policy. This will be analyzed in the next section.

4 Optimal Monetary Policy when Inflation Persistence is Random

Let the original model (4), (5) be assumed to fulfill the independence condition. After demeaning the variables as described in the previous section, we obtain the model (1), (2) with residuals given by (6) and (9) and covariances equaling zero, except $\sigma_{\alpha\eta^\pi} = \pi^* \cdot \sigma_\alpha^2$ and $\sigma_{\gamma\eta^y} = (\pi^* - \bar{i}) \cdot \sigma_\gamma^2$.

For deriving the optimal monetary policy rule, it is convenient to write the model in state-space representation:

(12) $\mathbf{x_{t+1}} = \mathbf{A_{t+1}} \cdot \mathbf{x_t} + \mathbf{b_{t+1}} \cdot \hat{i}_t + \mathbf{e_{t+1}}$

Here $\mathbf{x_{t+1}} = \begin{pmatrix} \hat{\pi}_{t+1} \\ y_{t+1} \end{pmatrix}$ is the state vector and $\hat{i}_t$ is the (scalar) control variable. Further, $\mathbf{A_{t+1}} = \begin{pmatrix} \alpha_{t+1} & \beta_{t+1} \\ \gamma_{t+1} & \delta_{t+1} \end{pmatrix}$ is an i.i.d. stochastic matrix with mean $\mathbf{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and diagonal covariance matrix $\mathbf{S_A}$, $\mathbf{b_{t+1}} = \begin{pmatrix} 0 \\ -\gamma_{t+1} \end{pmatrix}$ is an i.i.d. stochastic vector with mean $\mathbf{b} = \begin{pmatrix} 0 \\ -\gamma \end{pmatrix}$ and covariance matrix $\mathbf{S_b} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_\gamma^2 \end{pmatrix}$, and $\mathbf{e_{t+1}} = \begin{pmatrix} \eta_{t+1}^\pi \\ \eta_{t+1}^y \end{pmatrix}$ is an i.i.d. stochastic vector with mean $\mathbf{e} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and diagonal covariance matrix $\mathbf{S_e}$.

The central bank's optimizing problem can now be formulated as:

(13) $\min_{\{\hat{i}_\tau\}_{\tau=t}^\infty} E_t[\sum_{\tau=t}^\infty \omega^{\tau-t} \cdot \mathbf{x_\tau}'\mathbf{Q}\mathbf{x_\tau}]$

subject to (12). Here, $\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda_y \end{pmatrix}$ is the central bank's preference matrix. Defining the value function $J_t(\mathbf{x_t})$ as

(14) $J_t(\mathbf{x_t}) = \min_{\{\hat{i}_\tau\}_{\tau=t}^\infty} E_t[\sum_{\tau=t}^\infty \omega^{\tau-t} \cdot \mathbf{x_\tau}'\mathbf{Q}\mathbf{x_\tau}]$

we can formulate the following Bellman equation that is to be solved:

(15) $J_t(\mathbf{x_t}) = \min_{\hat{i}_t} \{\mathbf{x_t}'\mathbf{Q}\mathbf{x_t} + \omega \cdot E_t[J_{t+1}(\mathbf{x_{t+1}})]\}$

The solution of (15) can be obtained using the conjecture-and-verify approach. In contrast to the standard linear regulator problem under multiplicative uncertainty where the model's parameters and residuals are orthogonal, normalization has made the covariances $\sigma_{\alpha\eta^\pi}$ and $\sigma_{\gamma\eta^y}$ differ from zero and hence, the guess of the value function will need allowance for an additional first order term. This approach and the following solution method is general applicable to dynamic programming problems with non-orthogonal parameter uncertainty.

So we conjecture that the value function will take the form $J_t(\mathbf{x_t}) = \mathbf{x_t}'\mathbf{C_1}\mathbf{x_t} + \mathbf{c_2}'\mathbf{x_t} + c_3$, where $\mathbf{C_1}$ is a constant $2x2$-matrix, $\mathbf{c_2}$ is a constant vector and c_3 is a constant scalar. For zero covariances, the first order term $\mathbf{c_2}'$ will be the zero vector. It is shown in the appendix that inserting the guess into the Bellman equation (15) leads to:

$$
\begin{aligned}
& \mathbf{x_t}'\mathbf{C_1}\mathbf{x_t} + \mathbf{c_2}'\mathbf{x_t} + c_3 \\
& = \min_{\hat{i}_t} \Big\{ \mathbf{x_t}'\mathbf{Q}\mathbf{x_t} + \omega \cdot (\hat{i}_t \cdot (\mathbf{b}'\mathbf{C_1}\mathbf{b} + \mathbf{C_1}^{(22)} \cdot \sigma_\gamma^2) \cdot \hat{i}_t \\
& + \hat{i}_t \cdot \left[\mathbf{b}'(\mathbf{C_1} + \mathbf{C_1}')(\mathbf{A}\mathbf{x_t} + \mathbf{e}) - 2 \cdot \mathbf{C_1}^{(22)} \cdot \begin{pmatrix} \sigma_\gamma^2 \\ 0 \end{pmatrix}' \mathbf{x_t} - 2 \cdot \mathbf{C_1}^{(22)} \cdot \sigma_{\gamma\eta^y} \right] \\
& + (\mathbf{A}\mathbf{x_t} + \mathbf{e})'\mathbf{C_1}(\mathbf{A}\mathbf{x_t} + \mathbf{e}) \\
& + \mathbf{x_t}' \left[\mathbf{C_1}^{(11)} \begin{pmatrix} \sigma_\alpha^2 & 0 \\ 0 & \sigma_\beta^2 \end{pmatrix} + \mathbf{C_1}^{(22)} \begin{pmatrix} \sigma_\gamma^2 & 0 \\ 0 & \sigma_\delta^2 \end{pmatrix} \right] \mathbf{x_t} + \begin{pmatrix} 2 \cdot (\sigma_{\alpha\eta^\pi} \cdot \mathbf{C_1}^{(11)} + \sigma_{\gamma\eta^y} \cdot \mathbf{C_1}^{(22)}) \\ 0 \end{pmatrix}' \mathbf{x_t} \\
& + \mathbf{C_1}^{(22)} \cdot \sigma_{\eta^y}^2 + \mathbf{C_1}^{(11)} \cdot \sigma_{\eta^\pi}^2 + \mathbf{c_2}'(\mathbf{A} \cdot \mathbf{x_t} + \mathbf{b}\hat{i}_t + \mathbf{e}) + c_3) \Big\}
\end{aligned} \tag{16}
$$

Here, $\mathbf{C_1}^{(jk)}$ denotes the element of $\mathbf{C_1}$ in the j'th row and k'th column. From the right hand side of (16), the necessary first order condition for the optimization problem can be derived by differentiating with respect to $\hat{i}_t$ using the rules for the differentiating of quadratic and bilinear matrix forms (see for example Ljungqvist and Sargent, 2004, p. 110):

$$
\begin{aligned}
& (\mathbf{b}'(\mathbf{C_1} + \mathbf{C_1}')\mathbf{b} + 2 \cdot \mathbf{C_1}^{(22)} \cdot \sigma_\gamma^2)\hat{i}_t + \\
& \mathbf{b}'(\mathbf{C_1} + \mathbf{C_1}')(\mathbf{A}\mathbf{x_t} + \mathbf{e}) - 2 \cdot \mathbf{C_1}^{(22)} \begin{pmatrix} \sigma_\gamma^2 \\ 0 \end{pmatrix}' \mathbf{x_t} - 2 \cdot \mathbf{C_1}^{(22)} \cdot \sigma_{\gamma\eta^y} + \\
& \mathbf{c_2}'\mathbf{b} \overset{!}{=} 0
\end{aligned} \tag{17}
$$

So the optimal interest rate rule becomes:

$$
\begin{aligned}
& \hat{i}_t = (\mathbf{b}'(\mathbf{C_1} + \mathbf{C_1}')\mathbf{b} + 2 \cdot \mathbf{C_1}^{(22)} \cdot \sigma_\gamma^2)^{-1} \\
& \cdot ((2 \cdot \mathbf{C_1}^{(22)} \begin{pmatrix} \sigma_\gamma^2 \\ 0 \end{pmatrix}' - \mathbf{b}'(\mathbf{C_1} + \mathbf{C_1}')\mathbf{A})\mathbf{x_t} + (2 \cdot \mathbf{C_1}^{(22)} \cdot \sigma_{\gamma\eta^y}) - \mathbf{b}'(\mathbf{C_1} + \mathbf{C_1}')\mathbf{e} - \mathbf{c_2}'\mathbf{b}) \\
& = f_1 + \mathbf{f_2}'\mathbf{x_{t+1}}
\end{aligned} \tag{18}
$$

where

$$
f_1 = (\mathbf{b}'(\mathbf{C_1} + \mathbf{C_1}')\mathbf{b} + 2 \cdot \mathbf{C_1}^{(22)} \cdot \sigma_\gamma^2)^{-1} \cdot (2 \cdot \mathbf{C_1}^{(22)} \cdot \sigma_{\gamma\eta^y} - \mathbf{b}'(\mathbf{C_1} + \mathbf{C_1}')\mathbf{e} - \mathbf{c_2}'\mathbf{b}) \tag{19}
$$

and

$$(20)\quad \mathbf{f_2}' = (\mathbf{b}'(\mathbf{C_1}+\mathbf{C_1}')\mathbf{b} + 2\cdot\mathbf{C_1}^{(22)}\cdot\sigma_\gamma^2)^{-1}\cdot(2\cdot\mathbf{C_1}^{(22)}\begin{pmatrix}\sigma_\gamma^2\\0\end{pmatrix}' - \mathbf{b}'(\mathbf{C_1}+\mathbf{C_1}')\mathbf{A})$$

Labeling the first element of the vector $\mathbf{f_2}$ by f_π and the second element by f_y, allows a representation of the optimal feedback rule in the format of the Taylor (1993) rule:

$$(21)\quad \hat{i}_t = f_1 + f_\pi\cdot\hat{\pi}_t + f_y\cdot y_t$$

To solve for the unknown constants $\mathbf{C_1}$, $\mathbf{c_2}$ and c_3, one has to put the feedback rule (18) back into the Bellman equation (16). Rearranging gives:

(22)

$$\begin{aligned}
&\mathbf{x_t}'\mathbf{C_1}\mathbf{x_t} + \mathbf{c_2}'\mathbf{x_t} + c_3 = \\
&\mathbf{x_t}'\Bigg[\mathbf{Q} + \omega\,(\,\mathbf{f_2}(\mathbf{b}'\mathbf{C_1}\mathbf{b} + \mathbf{C_1}^{(22)}\cdot\sigma_\gamma^2)\mathbf{f_2}' + \\
&\mathbf{f_2}\,(\,\mathbf{b}(\mathbf{C_1}+\mathbf{C_1}')\mathbf{A} - 2\cdot\mathbf{C_1}^{(22)}\begin{pmatrix}\sigma_\gamma^2\\0\end{pmatrix}'\,) + \mathbf{A}'\mathbf{C_1}\mathbf{A} + \mathbf{C_1}^{(11)}\begin{pmatrix}\sigma_\alpha^2 & 0\\0 & \sigma_\beta^2\end{pmatrix} + \mathbf{C_1}^{(22)}\begin{pmatrix}\sigma_\gamma^2 & 0\\0 & \sigma_\delta^2\end{pmatrix}\,)\Bigg]\mathbf{x_t} + \\
&\omega\Bigg[f_1\cdot(\mathbf{b}'(\mathbf{C_1}+\mathbf{C_1}')\mathbf{b} + 2\cdot\mathbf{C_1}^{(22)}\cdot\sigma_\gamma^2)\mathbf{f_2}' + f_1(\mathbf{b}'(\mathbf{C_1}+\mathbf{C_1}')\mathbf{A} - 2\cdot\mathbf{C_1}^{(22)}\begin{pmatrix}\sigma_\gamma^2\\0\end{pmatrix}') + \\
&(\mathbf{e}'(\mathbf{C_1}+\mathbf{C_1}')\mathbf{b} - 2\cdot\mathbf{C_1}^{(22)}\cdot\sigma_{\gamma\eta^y})\mathbf{f_2}' + \mathbf{e}'(\mathbf{C_1}+\mathbf{C_1}')\mathbf{A} + \\
&\begin{pmatrix}2\cdot(\sigma_{\alpha\eta^\pi}\cdot\mathbf{C_1}^{(11)} + \sigma_{\gamma\eta^y}\cdot\mathbf{C_1}^{(22)})\\0\end{pmatrix}' + \mathbf{c_2}'(\mathbf{A} + \mathbf{b}\mathbf{f_2}')\Bigg]\mathbf{x_t} + \\
&\omega\cdot[\,f_1\cdot(\mathbf{b}'\mathbf{C_1}\mathbf{b} + \mathbf{C_1}^{(22)}\cdot\sigma_\gamma^2)\cdot f_1 + f_1\cdot(\mathbf{b}'(\mathbf{C_1}+\mathbf{C_1}')\mathbf{e} - 2\cdot\mathbf{C_1}^{(22)}\cdot\sigma_{\gamma\eta^y}) + \\
&\mathbf{e}'\mathbf{A}\mathbf{e} + \mathbf{C_1}^{(22)}\cdot\sigma_{\eta^y}^2 + \mathbf{C_1}^{(11)}\cdot\sigma_{\eta^\pi}^2 + \mathbf{c_2}'\mathbf{b}\cdot f_1 + \mathbf{c_2}'\mathbf{e} + c_3\,]
\end{aligned}$$

By comparison of coefficients, we have the following three equations which can be used step by step to calculate $\mathbf{C_1}$, $\mathbf{c_2}$ and c_3:

(23)

$$\mathbf{C_1} = \mathbf{Q} + \omega \Big[(\mathbf{f_2}(\mathbf{b}'\mathbf{C_1}\mathbf{b} + \mathbf{C_1}^{(22)} \cdot \sigma_\gamma^2)\mathbf{f_2}' + \mathbf{f_2}(\mathbf{b}(\mathbf{C_1} + \mathbf{C_1}')\mathbf{A} - 2 \cdot \mathbf{C_1}^{(11)} \begin{pmatrix} \sigma_\gamma^2 \\ 0 \end{pmatrix}') + \mathbf{A}'(\mathbf{C_1})\mathbf{A} + \mathbf{C_1}^{(11)} \begin{pmatrix} \sigma_\alpha^2 & 0 \\ 0 & \sigma_\beta^2 \end{pmatrix} + \mathbf{C_1}^{(22)} \begin{pmatrix} \sigma_\gamma^2 & 0 \\ 0 & \sigma_\delta^2 \end{pmatrix} \Big]$$

$$\mathbf{c_2}' = \omega \Big[f_1 \cdot [\mathbf{b}'(\mathbf{C_1} + \mathbf{C_1}')\mathbf{b} + 2 \cdot \mathbf{C_1}^{(22)} \cdot \sigma_\gamma^2]\mathbf{f_2}' + f_1[\mathbf{b}'(\mathbf{C_1} + \mathbf{C_1}')\mathbf{A} - 2 \cdot \mathbf{C_1}^{(22)} \begin{pmatrix} \sigma_\gamma^2 \\ 0 \end{pmatrix}'] + [\mathbf{e}'(\mathbf{C_1} + \mathbf{C_1}')\mathbf{b} - 2 \cdot \mathbf{C_1}^{(22)} \cdot \sigma_{\gamma\eta^y}]\mathbf{f_2}' + \mathbf{e}'(\mathbf{C_1} + \mathbf{C_1}')\mathbf{A} + \begin{pmatrix} 2 \cdot (\sigma_{\alpha\eta^\pi} \cdot \mathbf{C_1}^{(11)} + \sigma_{\gamma\eta^y} \cdot \mathbf{C_1}^{(22)}) \\ 0 \end{pmatrix}' + \mathbf{c_2}'(\mathbf{A} + \mathbf{b}\mathbf{f_2}') \Big] \quad (24)$$

$$c_3 = \omega \cdot \Big[f_1^2 \cdot (\mathbf{b}'\mathbf{C_1}\mathbf{b} + \mathbf{C_1}^{(22)}\sigma_\gamma^2) + f_1 \cdot [\mathbf{b}'(\mathbf{C_1} + \mathbf{C_1}')\mathbf{e} - 2 \cdot \mathbf{C_1}^{(22)} \cdot \sigma_{\gamma\eta^y}] + \mathbf{e}'\mathbf{A}\mathbf{e} + \mathbf{C_1}^{(22)} \cdot \sigma_{\eta^y}^2 + \mathbf{C_1}^{(11)} \cdot \sigma_{\eta^\pi}^2 + \mathbf{c_2}'\mathbf{b} \cdot f_1 + \mathbf{c_2}'\mathbf{e} + c_3 \Big] \quad (25)$$

Due to the non-linearity of the first two equations, an analytical solution is not available. Instead, the solution can be obtained by numerical methods for any given set of model parameters. First, iterating on (23) until convergence gives the matrix $\mathbf{C_1}$, which can be inserted into equation (24). Second, iterating on (24) until convergence leads to the vector $\mathbf{c_2}$. Finally, given $\mathbf{C_1}$ and $\mathbf{c_2}$, c_3 can be derived from (25).

Next, we compute the optimal policy interest rate rule $\hat{i}_t = f_1 + f_\pi \cdot \hat{\pi}_t + f_y \cdot y_t$ where there is only uncertainty about α_{t+1}, using the parameter setting of Söderström (2002), that is $\alpha = 1$, $\beta = 0.34$, $\gamma = 0.4$, $\delta = 0.77$, $\sigma_\alpha^2 = 0.1$, $\sigma_\beta^2 = \sigma_\gamma^2 = \sigma_\delta^2 = 0$ and $\omega = 0.99$ and letting the preference values λ_y vary from 0 to 2.[4] The long run averages π^* and $\bar{i}$ can be chosen arbitrarily (however, we should restrict them to take strictly positive values) but need to be fixed for the simulations. The following calculation is based on values $\pi^* = 2$ and $\bar{i} = 4$. The shock variances $\sigma_{\varepsilon^\pi}^2$ and $\sigma_{\varepsilon^y}^2$ are set to unity.

The resulting optimal interest rate rules are visualized by figures 1, 2 and 3. The first two figures show the reaction coefficients f_y (to the actual output gap) and f_π (to actual inflation). Both exhibit exactly the same properties as in Söderström (2002): They are declining with increasing weight on output stabilization λ_y and for any strictly positive λ_y, both reaction coefficients are higher under parameter uncertainty than under cer-

[4]Söderström chooses these parameter values on the basis of the euro-zone estimates of Orphanides and Wieland (2000).

tainty equivalence. The optimal central bank reaction to deviations from target is thus more aggressive when uncertainty about the dynamics of inflation is taken into account.

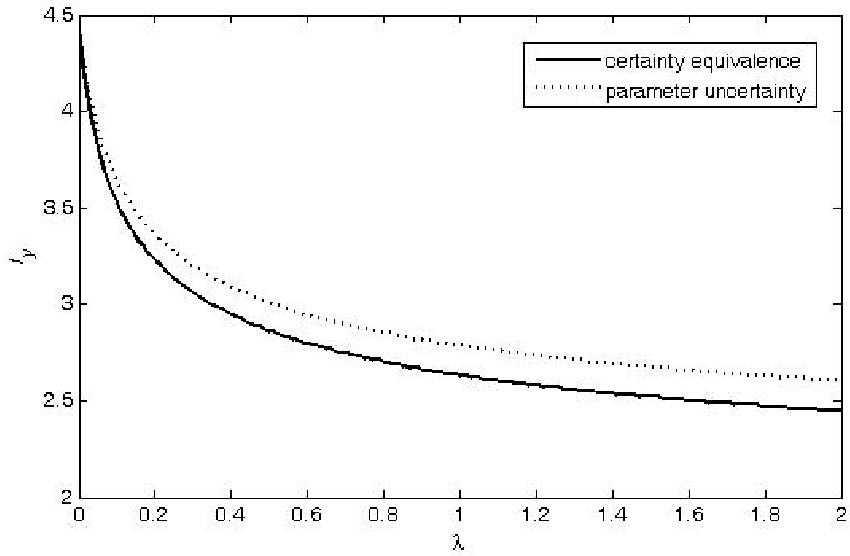

Figure 1: Reaction coefficient on current output gap

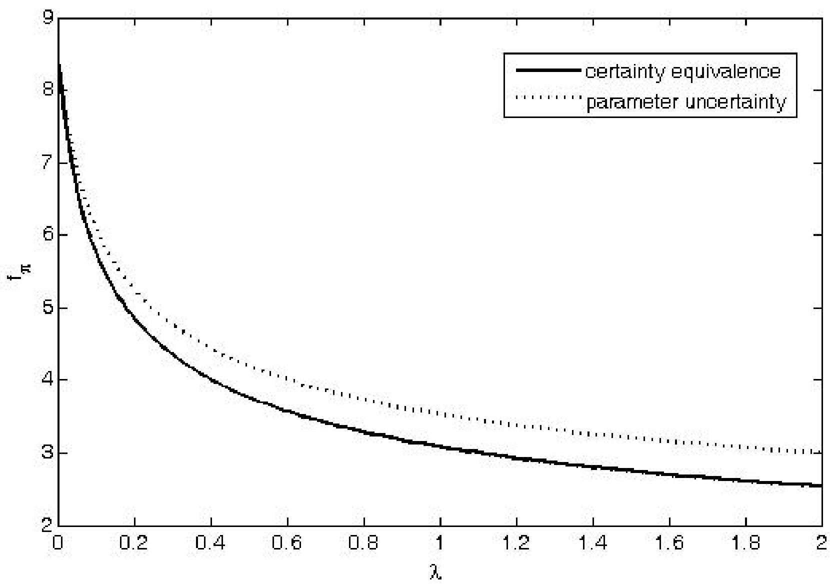

Figure 2: Reaction coefficient on actual inflation

Whereas under strict inflation targeting ($\lambda_y = 0$) the reaction coefficients remain certainty equivalent, the optimal feedback rule's intercept f_1 does not, as can be seen in figure 3. This intercept f_1 can be interpreted as a measure for the the optimal deviation of the neutral policy rate from the long run average $\bar{i}$. The neutral policy rate is

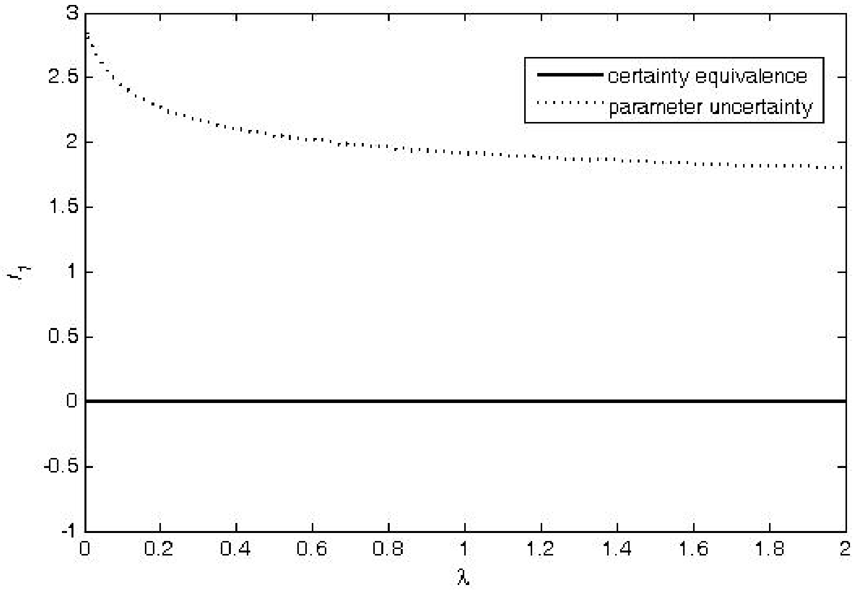

Figure 3: Neutral interest rate deviation from long run average

measured by the policy interest rate that is set when inflation and the output gap are at target. Under certainty equivalence, this rate equals the long run average $\bar{i}$, whereas uncertainty about the inflation persistence parameter α_{t+1} causes a positive deviation. In Söderström's setting, the neutral rate equals the long run average under parameter uncertainty as well. This can also be seen in equations (19) and (24) which determine the optimal neutral rate: If the covariances $\sigma_{\alpha\eta^\pi}$ and $\sigma_{\gamma\eta^y}$ are zero, both equations are simultaneously solved by $\mathbf{c_2} = \binom{0}{0}$ and $f_1 = 0$.

The positive deviation in our case is different from zero for all values of λ_y, especially for the strict inflation targeting case $\lambda_y = 0$. So in the case of strict inflation targeting, the central bank indeed reacts to deviations of inflation or output from target in the same way it would in a world of certainty, but these reactions take place on a higher average level. Concretely, with the chosen parameter setting and with $\lambda_y = 0$, the optimal interest rule under parameter uncertainty becomes $\hat{i}_t = 2.9 + 8.35 \cdot \hat{\pi}_t + 4.42 \cdot y_t$, so the neutral interest rate is 2.9 percentage points above the long run average $\bar{i}$. This result can be interpreted as larger aggressiveness in the sense of a more "hawkish" policy: For any given values of π_t and y_t, the central bank chooses a higher policy rate than under certainty equivalence, thus fighting inflation more strongly.

The reason is, that there is a positive covariance between the inflation persistence parameter and the additive inflation shock. This covariance increases with higher uncertainty and leads to a specific asymmetry with respect to additive inflation shocks: If a positive inflation shock moves inflation above target, inflation persistence will, on average, increase thus inflation is likely to stay away from target for a longer time. If a negative inflation shock decreases inflation below the target, inflation persistence will, on average, be lower, so the downwards-shock fades out more quickly than an upwards-shock. Hence, it is optimal to lean asymmetrically against shocks and thus setting interest rates on average higher than under certainty equivalence. This result reconciles the Söderström (2002) study with the findings by Craine (1979). The "hawkish" neutral stance under inflation persistence uncertainty corresponds to Craine's lower average money growth under uncertainty about the persistence of output.

5 Discussion and Application

In this section I will return to the question whether independence should be imposed before or after the model is normalized. Obviously, the answer is arbitrary. Both approaches just assume a certain covariance between the inflation persistence parameter and the additive inflation shock of the original model (4), (5): The first approach assumes that $\sigma_{\alpha\varepsilon^{\pi}} = 0$ and the second one implies $\sigma_{\alpha\varepsilon^{\pi}} = -\pi^{*} \cdot \sigma_{\alpha}^{2}$. However, the analysis has shown that the second assumption implies precisely that covariance that makes the long run average interest rate the optimal neutral one.

In fact, **any other value** for $\sigma_{\alpha\varepsilon^{\pi}}$ will lead to an optimal policy interest rate rule under parameter uncertainty that differs from the certainty equivalent case. This can be shown by calculating the coefficients for the optimal rule for different values of $\sigma_{\alpha\varepsilon^{\pi}}$. Although this argument is true for all values of λ_y, we will restrict our analysis to the case of strict inflation targeting. Figure 4 shows the optimal value of f_1 under uncertainty about α_{t+1} (the dotted line) in comparison with the horizontal zero-line that represents the optimal f_1 under certainty about α_{t+1}. λ_y is set to zero and $\sigma_{\alpha\varepsilon^{\pi}}$ increases from -0.31 to 0.31 (This interval covers the range for $\sigma_{\alpha\varepsilon^{\pi}}$ that is, given the variances $\sigma_{\alpha}^{2} = 0.1$ and $\sigma_{\varepsilon^{\pi}}^{2} = 1$, the interval $[-\sqrt{0.1}; +\sqrt{0.1}]$). Note that the coefficients f_{π} and f_y remain constant at their certainty equivalent values.

It can be seen, that f_1 increases monotonically with growing $\sigma_{\alpha\varepsilon^{\pi}}$ and that it is therefor different from zero except for the Söderström (2002) case, where $\sigma_{\alpha\varepsilon^{\pi}} = -\pi^{*} \cdot \sigma_{\alpha}^{2} = -0.2$. This case is marked by a vertical dashed line. A second vertical dashed line indicates the "independence before normalizing" case, where $\sigma_{\alpha\varepsilon^{\pi}} = 0$. The "hawkish" neutral monetary policy stance is thus representative for all values of $\sigma_{\alpha\varepsilon^{\pi}}$ higher than $-\pi^{*} \cdot \sigma_{\alpha}^{2}$. For all values of $\sigma_{\alpha\varepsilon^{\pi}}$ lower than $-\pi^{*} \cdot \sigma_{\alpha}^{2}$, optimal policy is less aggressive: The neutral policy rate is below the long run average $\bar{i}$, so that for any given values of π_t and y_t the central bank chooses a lower policy rate than under certainty equivalence. In this case it is optimal to fight inflation less aggressively - and parameter uncertainty

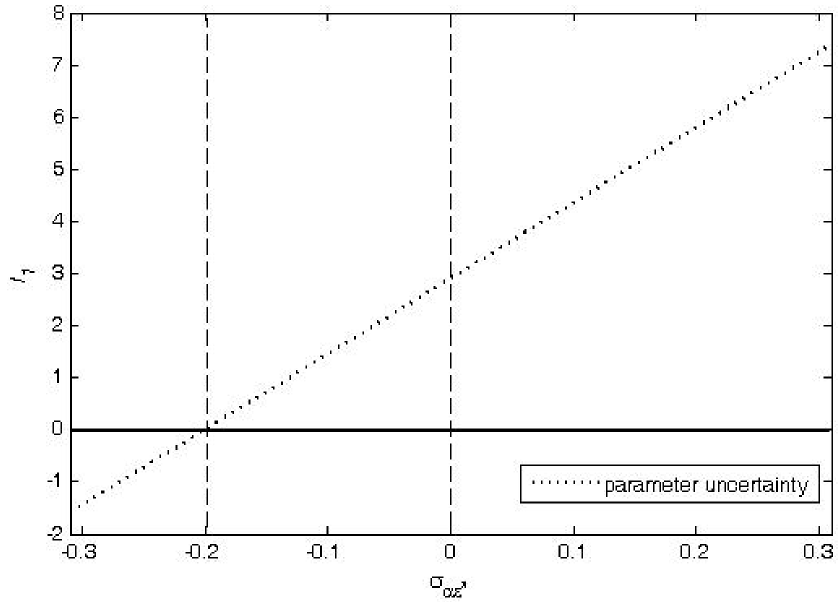

Figure 4: Neutral interest rate for different covariance values

supports the "doves".

We can draw the following conclusions: As soon as any independence assumption is made, a certain covariance between inflation persistence parameter and inflation shock is implied. In general, this covariance leads to optimal monetary policy under uncertainty about inflation persistence that is **not** certainty equivalent, even in the strict inflation targeting case. The optimal neutral stance of monetary policy differs from the long run average rate which is the optimal neutral interest rate under certainty equivalence. Thus, as long as the covariance is not assumed to equal exactly the only value that leads to certainty equivalence, uncertainty about the persistence of inflation may make the central bank fighting inflation more or less aggressively. The direction depends on the exact value of the covariance.

This result emphasizes the central importance of specifying the covariance. First, the value of the covariance should generally be rather determined empirically than set ad hoc. Second, even if the demeaned model (1), (2) with the independence property is regarded as adequate, any other normalization than demeaning will change the covariance between inflation persistence parameter and inflation shock and thus the implications of uncertainty on optimal monetary policy.

This provides a set of possible applications for the analysis conducted in this paper. For example, assume that the central bank wishes to change its inflation target as recently suggested by Blanchard et al. (2010). The authors recommend to rise the inflation target from its conventional value of 2 percent to 4 percent. This would enlarge the scope of monetary policy, because a higher inflation target should lead to a higher average nominal interest rate and thus provide the the central bank with more room for reactions to large economic shocks, like those that occurred during the financial crisis of 2007-2009.

In our framework, a regime shift from an inflation target of 2 percent to a higher inflation target of 4 percent can be modeled as follows: We start with an economy that is characterized by equations (1) and (2) where inflation is normalized by subtracting the old inflation target of 2 percent and the nominal interest rate is normalized by subtraction of the long run average value of 4 percent. The model is assumed to fulfill the independence condition. Now, the central bank raises it's inflation target to the value of 4 percent. To compute the new optimal policy rule, the inflation variable is transformed by a further subtraction of 2 percentage points while the nominal interest rate variable remains unchanged. This procedure technically corresponds to setting up model (4), (5) with $\varepsilon^{\pi} = \varepsilon^{y} = 0$ and independence property and then normalizing it as in section 4, but with $\pi^{*} = 2$ and $\bar{i} = 0$. As this new normalization increases the covariance between inflation persistence parameter and inflation shock, a higher neutral stance than under certainty equivalence is to be expected. Calculating the optimal policy interest rate rule for the higher inflation target gives the same reaction coefficients as before (see figures 1 and 2) and a neutral policy rate which is parallel shifted upwards by 2 percentage points as it is illustrated in figure 5.

Whereas certainty equivalence indicates a one-by-one rise of the neutral rate with the rise of the inflation target, inflation persistence uncertainty leads to an even higher increase. Again, this is also true for the case of strict inflation targeting which even shows the strongest effect. The analysis thus suggests that the monetary policy scope Blanchard and his co-authors hope to gain will even be greater if parameter uncertainty is taken into account. There are good arguments for assuming that this effect is not only of theoretical nature. As Altissimo et al. (2006) summarize, inflation persistence is substantially influenced by the anchoring of inflation expectations. Further, empirical research indicates that explicit inflation targets help to anchor inflation expectations (see for example Gurkaynak et al., 2006). Hence, a regime shift toward a higher inflation target may disturb the anchoring of inflation expectations, and thus, increase the uncertainty about the persistence of inflation.

There are, of course, also possible situations with opposing effect. Suppose, for example, the central bank considers the historical average inflation rate as being too high and announces an inflation target below this value. The reduction of the inflation target will lead to a decreasing covariance between inflation persistence parameter and inflation shock and hence increasing parameter uncertainty may lead to a lower

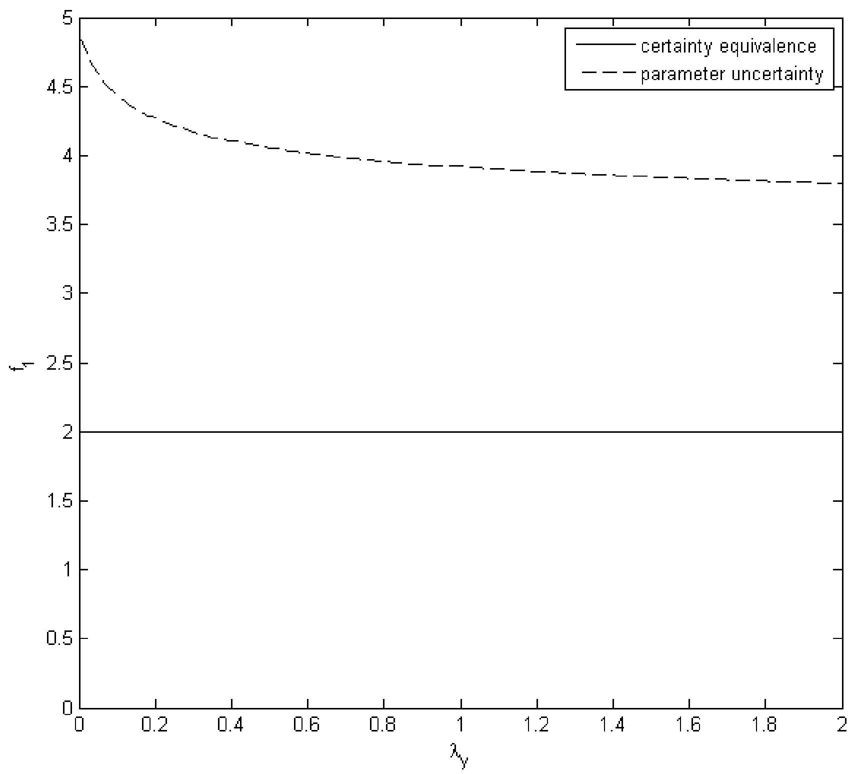

Figure 5: Neutral interest rate under a high inflation targeting regime

neutral stance than under certainty equivalence. We thus end up with an astonishing cross-effect of inflation persistence uncertainty: If the monetary policy regime is shifted toward a looser policy, it backs up the "hawks", but for any change in direction of a tighter regime, it gives support to the "doves".

6 Concluding Remarks

This paper starts by indicating an apparent contradiction in the literature on the impact of inflation persistence uncertainty on optimal monetary policy in the case of strict inflation targeting. On the one hand, the findings by Craine (1979) imply greater aggression, whereas Söderström (2002) finds certainty equivalence. These results are reconciled by analyzing the deeper interrelations between normalization and independence assumptions. It is shown, that the Söderström setup implies a certain condition for the covariance between inflation persistence parameter and inflation shock, which leads to the certainty equivalence result. However, any other value for the covariance causes the optimal neutral policy interest rate to deviate from the long run average policy rate. This result brings the Söderström analysis again in line with Craine's and is not restricted to the strict inflation targeting case. To emphasize the importance of this effect on the optimal neutral stance, the modeling framework is applied to the recent Blanchard et al. (2010) proposal, that advocates an increase in the inflation target from 2 to 4 percent. It is shown that this target shift in combination with inflation persistence uncertainty leads to a higher neutral monetary policy stance than under certainty equivalence. Conversely, for any reduction of the inflation target, inflation persistence uncertainty would support a lower neutral stance.

It can be summarized that uncertainty about the persistence of inflation makes optimal monetary policy in general not certainty equivalent and causes interesting diametrical effects on the optimal neutral stance of monetary policy in the case of a change in the inflation target: For any shift toward a tighter regime, uncertainty supports the "doves" in the central bank's decision committee, while for any loosening in the inflation target it is on the side of the "hawks".

Appendix:

Derivation of equation (16):

Inserting the conjecture $J_t(\mathbf{x_t}) = \mathbf{x_t}'\mathbf{C_1}\mathbf{x_t} + \mathbf{c_2}'\mathbf{x_t} + c_3$ into the Bellman equation (15) and applying the rule for the expectation of quadratic matrix forms leads to:

$$
\begin{aligned}
&\mathbf{x_t}'\mathbf{C_1}\mathbf{x_t} + \mathbf{c_2}'\mathbf{x_t} + c_3 \\
&= \min_{\hat{i}_t} \left\{ \mathbf{x_t}'\mathbf{Q}\mathbf{x_t} + \omega \cdot E_t[\mathbf{x_{t+1}}'\mathbf{C_1}\mathbf{x_{t+1}} + \mathbf{c_2}'\mathbf{x_{t+1}} + c_3] \right\} \\
&= \min_{\hat{i}_t} \left\{ \mathbf{x_t}'\mathbf{Q}\mathbf{x_t} + \omega \cdot \left(E_t[\mathbf{x_{t+1}}]'\mathbf{C_1} E_t[\mathbf{x_{t+1}}] + tr[\mathbf{C_1} \cdot \mathbf{S_{x_{t+1}}}] + \mathbf{c_2}' E_t[\mathbf{x_{t+1}}] + c_3 \right) \right\} \\
&= \min_{\hat{i}_t} \{ \mathbf{x_t}'\mathbf{Q}\mathbf{x_t} + \omega \cdot ((\mathbf{A} \cdot \mathbf{x_t} + \mathbf{b}\hat{i}_t + \mathbf{e})'\mathbf{C_1}(\mathbf{A} \cdot \mathbf{x_t} + \mathbf{b}\hat{i}_t + \mathbf{e}) + tr[\mathbf{C_1} \cdot \mathbf{S_{x_{t+1}}}] + \\
&\mathbf{c_2}'(\mathbf{A} \cdot \mathbf{x_t} + \mathbf{b}\hat{i}_t + \mathbf{e}) + c_3) \} \\
&= \min_{\hat{i}_t} \{ \mathbf{x_t}'\mathbf{Q}\mathbf{x_t} + \omega \cdot (\hat{i}_t \cdot (\mathbf{b}'\mathbf{C_1}\mathbf{b} + \mathbf{C_1}^{(22)} \cdot \sigma_\gamma^2) \cdot \hat{i}_t \\
&+ \hat{i}_t \cdot (\mathbf{b}'(\mathbf{C_1} + \mathbf{C_1}')(\mathbf{A}\mathbf{x_t} + \mathbf{e}) - 2 \cdot \mathbf{C_1}^{(22)} \cdot \begin{pmatrix} \sigma_\gamma^2 \\ 0 \end{pmatrix}' \mathbf{x_t} - 2 \cdot \mathbf{C_1}^{(22)} \cdot \sigma_{\gamma\eta^y}) \\
&+ (\mathbf{A}\mathbf{x_t} + \mathbf{e})'\mathbf{C_1}(\mathbf{A}\mathbf{x_t} + \mathbf{e}) \\
&+ \mathbf{x_t}'(\mathbf{C_1}^{(11)} \begin{pmatrix} \sigma_\alpha^2 & 0 \\ 0 & \sigma_\beta^2 \end{pmatrix} + \mathbf{C_1}^{(22)} \begin{pmatrix} \sigma_\gamma^2 & 0 \\ 0 & \sigma_\delta^2 \end{pmatrix})\mathbf{x_t} + \begin{pmatrix} 2 \cdot (\sigma_{\alpha\eta^\pi} \cdot \mathbf{C_1}^{(11)} + \sigma_{\gamma\eta^y} \cdot \mathbf{C_1}^{(22)}) \\ 0 \end{pmatrix}' \mathbf{x_t} \\
&+ \mathbf{C_1}^{(22)} \cdot \sigma_{\eta^y}^2 + \mathbf{C_1}^{(11)} \cdot \sigma_{\eta^\pi}^2 + \mathbf{c_2}'(\mathbf{A} \cdot \mathbf{x_t} + \mathbf{b}\hat{i}_t + \mathbf{e}) + c_3) \}
\end{aligned}
$$

Note that $tr[\cdot]$ is the trace-operator, $\mathbf{S_{x_{t+1}}}$ is the conditional covariance matrix of $\mathbf{x_{t+1}}$ and $\mathbf{C_1}^{(jk)}$ denotes the element of $\mathbf{C_1}$ in the j'th row and k'th column. In the last step, we use the fact that the covariance between $\hat{\pi}_{t+1}$ and y_{t+1} is zero. Thus, the trace can be calculated as:

$$
\begin{aligned}
tr[\mathbf{C_1 S_{x_{t+1}}}] &= \mathbf{C_1}^{(11)} \cdot Var_t[\hat{\pi}_{t+1}] + \mathbf{C_1}^{(22)} \cdot Var_t[y_{t+1}] \\
&= \mathbf{C_1}^{(11)} \cdot Var_t[\alpha_{t+1}\hat{\pi}_t + \beta_{t+1} y_t + \eta^{\pi}_{t+1}] \\
&+ \mathbf{C_1}^{(22)} \cdot Var_t[\delta_{t+1} y_t - \gamma_{t+1}\hat{i}_t + \gamma_{t+1}\hat{\pi}_t + \eta^{y}_{t+1}] \\
&= \mathbf{C_1}^{(11)} \cdot [\hat{\pi}_t^2 \cdot \sigma^2_{\alpha} + y_t^2 \cdot \sigma^2_{\beta} + \sigma^2_{\eta^{\pi}} + 2 \cdot \sigma_{\alpha\eta^{\pi}}] \\
&+ \mathbf{C_1}^{(22)} \cdot [y_t^2 \cdot \sigma^2_{\delta} + \hat{i}_t^2 \cdot \sigma^2_{\gamma} + \hat{\pi}_t^2 \cdot \sigma^2_{\gamma} + \sigma^2_{\eta^{y}} \\
&- 2 \cdot \hat{i}_t \cdot \hat{\pi}_t \cdot \sigma^2_{\gamma} - 2 \cdot \hat{i}_t \cdot \sigma_{\gamma\eta^{y}} + 2 \cdot \hat{\pi}_t \cdot \sigma_{\gamma\eta^{y}}] \\
&= \mathbf{x_t}'[\mathbf{C_1}^{(11)} \begin{pmatrix} \sigma^2_{\alpha} & 0 \\ 0 & \sigma^2_{\beta} \end{pmatrix} + \mathbf{C_1}^{(22)} \begin{pmatrix} \sigma^2_{\gamma} & 0 \\ 0 & \sigma^2_{\delta} \end{pmatrix}]\mathbf{x_t} + \begin{pmatrix} 2 \cdot (\mathbf{C_1}^{(11)} \cdot \sigma_{\alpha\eta^{\pi}} + \mathbf{C_1}^{(22)} \cdot \sigma_{\gamma\eta^{y}}) \\ 0 \end{pmatrix}' \mathbf{x_t} \\
&- 2 \cdot \mathbf{C_1}^{(22)} \cdot \hat{i}_t \cdot \begin{pmatrix} \sigma^2_{\gamma} \\ 0 \end{pmatrix}' \mathbf{x_t} \\
&+ \hat{i}_t \cdot \mathbf{C_1}^{(22)} \cdot \sigma^2_{\gamma} \cdot \hat{i}_t - 2 \cdot \mathbf{C_1}^{(22)} \sigma_{\gamma\eta^{y}} \cdot \hat{i}_t + \mathbf{C_1}^{(22)} \cdot \sigma^2_{\eta^{y}} + \mathbf{C_1}^{(11)} \cdot \sigma^2_{\eta^{\pi}}
\end{aligned}
$$

Here, $Var_t[\cdot]$ denotes the conditional variance-operator. Expanding the quadratic matrix form $(\mathbf{A} \cdot \mathbf{x_t} + \mathbf{b}\hat{i}_t + \mathbf{e})'\mathbf{C_1}(\mathbf{A} \cdot \mathbf{x_t} + \mathbf{b}\hat{i}_t + \mathbf{e})$ and rearranging gives the right hand side of equation (16).

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