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Price and Inventory Dynamics in an Oligopoly Industry: A Framework for Commodity Markets

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Price and Inventory Dynamics in an Oligopoly Industry: A Framework for Commodity Markets*

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Abstract

This paper analyzes the interaction between price and inventory decisions in an oligopoly industry and its implications for the dynamics of prices. The work extends existing literature and especially the work of Hall and Rust (2007) to endogenous prices and strategic oligopoly competition. We show that the optimal decision rule is an (S, s) order policy and prices and inventory are strategic substitutes. Fixed ordering costs generate infrequent orders. Consequently, with strategic competition in prices, (S, s) inventory behavior together with demand uncertainty generates endogenous cyclical patterns in prices without any exogenous shocks. Hence, the developed model provides a promising framework for explaining dynamics of commodity markets and especially observed autocorrelation in price fluctuations.

Keywords: Inventory dynamics, price competition, oligopoly, (S, s) order policy, commodity markets.

JEL classification numbers: D21, D43, E22, L81.

1 Introduction

This paper analyzes the interaction between price and inventory decisions in an oligopoly industry and its implications for the dynamics of prices such as price dispersion. Cross-sectional price dispersion is a common feature in many retail markets. Since Stigler's (1961) seminal work price dispersion has usually been explained by consumer search

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costs. In contrast, Aguirregabiria (1999) shows that retail inventories can generate (S, s) dynamics of inventories which in turn can explain time variability of prices of super-market chains.[1] However, as in his model monopolistic competition is analyzed price dispersion between different firms can not be observed.

Extending the described work, this paper addresses the question how oligopolistic competition affects these dynamics.[2]

Previous papers have characterized the optimal decision rules of similar dynamic models. In addition to Aguirregabiria (1999) who analyzes price and inventories with lump-sum costs under monopolistic competition Hall and Rust (2007) study optimal inventory decisions with lump-sum costs under perfect competition. Their paper extends the framework of Aguirregabiria (1999) in some ways but is otherwise limited to one decision variable as prices are taken as given. Hall and Rust (2007) show that in their perfect competition model the (S, s) policy is an optimal order strategy.[3] To the best of our knowledge, these two works studying extreme cases of competition are by far the most elaborated papers investigating these decision problems.[4] The analysis of optimal decision rules under oligopolistic competition forms an obvious gap in the literature.

However, related studies of oligopolistic competition exists. Dutta and Sundaram (1992) and Dutta and Rustichini (1995) analyze a discrete choice stochastic duopoly game with lump-sum costs. In these frameworks the one abstract decision variable affecting both firms' payoffs cannot be interpreted as being related to inventory. Nevertheless, the optimality of an (S, s) policy can also be shown. More recently, Besanko and Doraszelski (2004) study decisions about prices and capacity. However, the main and important difference between inventory and capacity is that excess capacity is worthless while keeping inventory affects future competition. Hence, additional strategic effects due to kept stock are at place. This is especially important when investigating oligopolistic competition.

This paper extends the literature by characterizing an equilibrium in a model of price and inventory competition in oligopoly. We allow oligopolistic firms to interact strategically. This allows for studying price dispersion between firms.

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[1] Under an (S, s) rule inventory moves between the target inventory level, S, and the order threshold, s, with $s < S$. Whenever the firm's inventory level falls below the order threshold, a new order is placed such that the target inventory level S is attained.

[2] Additionally, the main focus of the paper by Aguirregabiria is an empirical analysis building on a numeric simulation. The formal theoretical proof of the optimality of the considered inventory decision is therefore not rigourously done and incomplete. Thus, our paper is the first to formally prove the optimality of (S, s) policy with endogenous prices.

[3] Thereby Hall and Rust (2007) extend earlier work like Sethi and Cheng (1997) and Cheng and Sethi (1999) to a more general specification of the Markov process.

[4] There exist also some papers analyzing dynamic oligopoly with inventories without considering lump-sum ordering cost, like Kirman and Sobel (1974) or more recently Bernstein and Federgruen (2004). However, without ordering cost stationary optimal strategies result which are in essence identical to those of the corresponding static single period game.

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Besides, such a model that is incorporating inventory and oligopoly in dynamic competition provides the most plausible framework for retail industries. Retail industries have become highly concentrated, i.e., in most categories like grocery, supermarkets, and office supplies just a handful of rivals compete locally. In the supermarket industry for example a small number of firms capture the majority of sales as supermarkets compete in tight regional oligopolies. Thus, this industry is a prime example of oligopoly. Besides, inventory costs are of major importance. Supermarkets invest in state of the art distribution systems to minimize storage and transportation costs (see e.g. Beresteanu & Ellickson, 2006; Ellickson, 2007). Hence, deciding the optimal inventory and store offer forms an important optimization problem for supermarket chains.

In this work we study the decision problem of a central store, i.e., its decision about retail prices and orders to suppliers, facing oligopolistic competition and taking into account the existence of lump-sum ordering cost. We develop a model of retail competition in which the impact of inventories on competition and prices can be evaluated. We analyze the characteristics of the optimal decision rule.

The main findings of our theoretical model of oligopoly support the simulation results of Aguirregabiria (1999) studying monopoly. Key factors for price fluctuations are lump-sum ordering costs and demand uncertainty. Lump-sum ordering cost generate (S, s) inventory behavior. Demand uncertainty creates a positive probability of excess demand, i.e., stockouts. The positive stockout probability has a negative effect on expected sales which in turn creates substitutability between prices and inventories in the profit function such that in equilibrium prices depend negatively and very significantly on the level of inventories. This results in a cyclical pattern of inventories and prices where prices decline significantly when an order is placed and consequently inventory reduction generates price increase. The pricing behavior in this model can generate cross-sectional price dispersion with cyclical patterns even without menu costs.

The rest of the paper is organized as follows. Section 2 introduces the model and shows important characteristics of firms' expected sales. Section 3 characterizes the optimal decision rules. Section 4 concludes while the Appendix contains the proofs of the results stated in the text.

2   The Model

Consider an oligopoly market where risk neutral firms, indexed by $i \in \{1, 2, ..., N\}$, sell differentiated storable products. Each firm sells a variety of the product. Firms compete in prices and they have uncertainty about temporary demand shocks. In the short run, firms cannot respond to these temporary shocks neither by changing prices nor by increasing supply, in case of excess demand. Firms do not face any delivery lags and cannot backlog unfilled orders. Thus, whenever demand exceeds quantity on hand,

the residual unfilled demand is lost. Therefore, the quantity sold by firm i at period t is the minimum of supply and demand:

$$y_{it} = \min\{s_{it} + q_{it}, d_{it}\}, \tag{1}$$

where y_{it} is the quantity sold; s_{it} is the level of inventories at the beginning of period t; q_{it} represents new orders to wholesalers during period t; and d_{it} is consumers' demand. Every period t a firm knows the levels of inventories of all the firms in the market, i.e., the vector $\mathbf{s}_t \equiv \{s_{1t}, s_{2t}, ..., s_{Nt}\}$.[5] Given this information, the firm decides on prices and new orders (p_{it}, q_{it}) to maximize its expected value $E_t(\sum_{r=1}^{\infty} \beta^r \Pi_{i,t+r})$, where $\beta \in (0,1)$ is the discount factor and Π_{it} is the current profit of firm i at period t.

A firm's current profit is equal to revenue minus ordering cost and inventory holding cost:

$$\Pi_{it} = p_{it} y_{it} - c_i q_{it} - k_i I\{q_{it} > 0\} - h_i s_{it}, \tag{2}$$

where c_i is the unit ordering cost; k_i is the fixed or lump-sum ordering cost; and h_i is the inventory holding cost.

The transition rule of inventories, i.e., state variables, is:

$$s_{it+1} = s_{it} + q_{it} - y_{it} = \max\{0, s_{it} + q_{it} - d_{it}\}. \tag{3}$$

Firms have uncertainty about current demand. The demand of product i at period t is

$$d_{it} = \exp\{\varepsilon_{it}\} d_{it}^e.$$

Here, ε_{it} is a temporary and idiosyncratic demand shock that is independently and identically distributed over time with cumulative distribution function $F(\cdot)$ that is continuously differentiable on the Lebesgue measure. These shocks are unknown to firms when they decide prices and orders. Furthermore, d_{it}^e is the expected demand that depends on the endogenous prices and the exogenous qualities of all products. The expected demand d_{it}^e is a function of the prices of all firms such that it is strictly increasing in the own price, strictly decreasing in the prices of competitors, and the revenue function $p_i d_i^e$ is strictly concave in p_i. By definition of expected demand, we have that $E(\exp\{\varepsilon_{it}\}) = 1$. For technical reasons it is useful to assume that $F(\cdot)$ is such that the respective hazard rate $h(\cdot) = \frac{f(\cdot)}{1-F(\cdot)}$ is smaller than one.[6] For examples and numerical exercises it may be useful to consider a logit demand model for the expected demand:

$$d_{it}^e = \frac{\exp\{w_i - \alpha p_{it}\}}{1 + \sum_{j=1}^{N} \exp\{w_j - \alpha p_{jt}\}}, \tag{4}$$

where $\{w_i : i = 1, 2, ..., N\}$ are exogenous parameters that represent product qualities, and α is a parameter that represents the marginal utility of income. The logit demand

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[5]This is a very reasonable assumption as firms can observe prices and are therefore able to learn and deduce stock levels.

[6]This assumption is especially helpful for proving Lemma 2, although it is only a sufficient but not necessary condition.

model is convenient for the derivation and illustration of some future results, but it can be relaxed for all our results.[7]

2.1 Implications of Demand Uncertainty for Expected Sales

As a firm does not know the temporary demand shock ε_{it}, it does not know actual sales y_{it}. Expected profits are $\Pi_{it}^e = p_{it} \, y_{it}^e - c_i \, q_{it} - k_i \, I\{q_{it} > 0\} - h_i s_{it}$, where y_{it}^e represents expected sales, i.e., $y_{it}^e = E[\min\{d_{it}, s_{it} + q_{it}\}]$. Demand uncertainty has important implications for the relationship between prices and inventories.

Lemma 1. *Expected sales y_{it}^e are equal to expected demand d_{it}^e times a function $\lambda\left(\frac{s_{it}+q_{it}}{d_{it}^e}\right)$, i.e.,*

$$y_{it}^e = d_{it}^e \lambda\left(\frac{s_{it} + q_{it}}{d_{it}^e}\right). \tag{5}$$

The function $\lambda(x)$ is defined as $\int \min\{x, \exp(\varepsilon)\}dF(\varepsilon)$ and it has the following properties:

(i) It is continuously differentiable;

(ii) it is strictly increasing;

(iii) $\lambda(0) = 0$;

(iv) $\lambda(\infty) = E(\exp(\varepsilon)) = 1$; and

(v) for $x > 0$, $\lambda'(x) = \int_{-\infty}^{\ln(x)} dF(\varepsilon) = 1 - F(\ln(x)) \in (0, 1)$.

Proof: See Appendix A.1.

In case of a very small (close to zero) supply-to-expected-demand-ratio $\frac{s_{it}+q_{it}}{d_{it}^e}$ stockout probability is very large such that expected sales are much lower than expected demand (approaching zero). On the other hand, a high ratio (approaching infinity) yields low probability for stockouts such that expected sales are almost equal to expected demand. The higher the supply-to-expected-demand-ratio the lower gets the probability of stockout and the more do expected sales converge to expected demand. This is formalized in properties *(ii)* - *(iv)*. From property *(v)* yielding $\lambda''(x) < 0$ it is now clear that the gain of a higher supply-to-expected-demand-ratio for expected sales is higher the lower the ratio. For low ratios the gain is almost equal to the increase of stock as one unit more in stock in essence is a unit more sold. For high ratios the probability of selling an additional unit in stock decreases to zero.

Therefore, variability over time in the supply-to-expected-demand-ratio can generate significant fluctuations in expected sales and thus in optimal prices.

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[7]See Aguirregabiria (2007) for a derivation of this demand model from a model of consumer behavior under possible excess demand.

2.2 Markov Perfect Equilibrium

The model has a Markov structure and we assume that firms play Markov strategies. That is, a firm's strategy depends only on payoff relevant state variables, which in this model is the vector of inventories $\mathbf{s}_t$. Therefore, a strategy for firm i is a function $\sigma_i(\mathbf{s}_t)$ from the space of the vector of inventories, $\mathbb{R}_+^N$, into the space of the decision variables (p_{it}, q_{it}), $\mathbb{R}_+^2$, i.e., $\sigma_i(\mathbf{s}_t)$ is a function from $\mathbb{R}_+^N$ into $\mathbb{R}_+^2$. Let $\sigma \equiv \{\sigma_i : i = 1, 2, ..., N\}$ be a set of strategy functions, one for each firm. Suppose that firm i considers the rest of the firms to behave according to their respective strategies in σ. Under this condition, other firms' inventories, $\mathbf{s}_{-it}$, follow a Markov transition probability function $F_{\mathbf{s}_{-i}}^\sigma(\mathbf{s}_{-it+1}|\mathbf{s}_{-it})$. Note that this transition probability function depends on the other firms' strategies in σ. Taking $F_{\mathbf{s}_{-i}}^\sigma$ as given, firm i's decision problem can be represented using the Bellman equation:

$$V_i^\sigma(\mathbf{s}_t) = \max_{\{p_i, q_i\}} \left\{ \Pi_i^\sigma(p_i, s_{it} + q_i) + \beta \int V_i^\sigma(s_{i,t+1}, \mathbf{s}_{-it+1}) dF(\varepsilon_{it}) dF_{\mathbf{s}_{-i}}^\sigma(\mathbf{s}_{-it+1}|\mathbf{s}_{-it}) \right\}.$$
(6)

The (expected) profit function is continuously differentiable and the standard regularity conditions apply such that the value function V_i^σ is uniquely determined as the fixed point of a contraction mapping. Note that this value function is conditional to the other firms' strategies. A Markov perfect equilibrium (MPE) is a set of equilibrium strategies σ such that for every firm i and for every vector $\mathbf{s}_t \in \mathbb{R}_+^N$ we have that

$$\sigma_i(\mathbf{s}_t) = \arg \max_{\{p_i, q_i\}} \left\{ \Pi_i^\sigma(p_i, s_{it} + q_i) \right.$$
$$\left. + \beta \int V_i^\sigma(s_{i,t+1}, \mathbf{s}_{-it+1}) dF(\varepsilon_{it}) dF_{\mathbf{s}_{-i}}^\sigma(\mathbf{s}_{-it+1}|\mathbf{s}_{-it}) \right\}.$$
(7)

3 Optimal Decision Rule

Let us now characterize the optimal decision rule for a firm in this game of oligopolistic competition.

In this section we will show that the (S, s) rule is indeed the best response not only to an (S, s) rule but to any given strategy of the opponents. This, of course, implies that the equilibrium resulting from (S, s) strategies by all players is a MPE.

In order to represent the optimal decision rule of the oligopolists, it is convenient to represent the decision problem in terms of the variables p_{it} and $z_{it} \equiv s_{it} + q_{it}$. The variable z_{it} represents the total supply of the product during period t. It is also useful to define the following "value" function which is independent of the firm's own current inventory, i.e., the only state variable the firm can influence (however, it is not independent of the

current state per se), and taking the other firms' strategies in σ and so $F^\sigma_{\mathbf{s}_{-i}}$ as given:

$$Q^\sigma_i(z_{it}, p_{it}; \mathbf{s}_{-it}) \equiv -cz_{it} + p_{it} \int \min\left\{z_{it}; e^{\varepsilon_{it}} d^{e\sigma}_{it}(p_{it})\right\} dF(\varepsilon_{it})$$

$$+ \beta \int V^\sigma_i \left(\max\left\{0; z_{it} - e^{\varepsilon_{it}} d^{e\sigma}_{it}(p_{it})\right\}; \mathbf{s}_{-it+1}\right) dF(\varepsilon_{it}) dF^\sigma_{\mathbf{s}_{-i}}(\mathbf{s}_{-it+1}|\mathbf{s}_{-it}) \quad (8)$$

such that

$$V^\sigma_i(\mathbf{s}_t) = \max_{\{p_i, q_i\}} \left\{Q^\sigma_i(s_{it} + q_i, p_i; \mathbf{s}_{-it}) - (h_i - c_i)s_{it} - k_i I_{\{q_i > 0\}}\right\}.$$

Given the function Q^σ_i, it is clear that an oligopolist chooses (z_{it}, p_{it}) as a best response to the other firms' strategies in σ, i.e., other firms order and pricing decisions, to maximize $Q^\sigma_i(z_{it}, p_{it}; \mathbf{s}_{-it}) - kI\{z_{it} > s_{it}\}$. Making use of this "value" function Q^σ_i we can derive important characteristics of competition in prices and inventories:

Lemma 2. *The function Q^σ_i is such that:*

(i) Q^σ_i is strictly concave in prices, i.e., $\partial^2 Q^\sigma_i(z_i, p_i)/\partial p_i \partial p_i < 0$.

(ii) Prices and total supply are strategic substitutes, i.e., $\partial^2 Q^\sigma_i(z_i, p_i)/\partial p_i \partial z_i \leq 0$.

Proof: See Appendix A.2.

The positive stockout probability has a negative effect on expected sales which in turn creates substitutability between prices and inventories in the profit function. This is the case as with low inventory optimal expected demand (under given demand uncertainty) is low and thus optimal price is high.

Using $\sigma^\sigma_p(\mathbf{s})$ and $\sigma^\sigma_z(\mathbf{s})$ to represent the optimal response rules for p and z, respectively, we have

$$\left\{\sigma^\sigma_{iz}(\mathbf{s}), \sigma^\sigma_{ip}(\mathbf{s})\right\} = \arg\max_{\{z_i \geq s_i, p_i \geq 0\}} \left\{Q^\sigma_i(z_i, p_i; \mathbf{s}_{-it}) - kI\{z_i > s_i\}\right\}.$$

We define the optimal price as a function of current supply:

$$\bar{p}^\sigma_i(z_i; \mathbf{s}_{-i}) \equiv \arg\max_{\{p_i\}} Q^\sigma_i(z_i, p_i; \mathbf{s}_{-i}). \quad (9)$$

Since Q^σ_i is continuously differentiable and strictly concave in prices, $\bar{p}^\sigma_i(z; \mathbf{s}_{-i})$ is implicitly defined by the first order condition $\frac{\partial Q^\sigma_i(z_i, \bar{p}_i; \mathbf{s}_{-i})}{\partial \bar{p}_i} = 0$.

It is now possible to show that the best response to any strategy is an (S, s) rule:

Proposition 1. *Firm i considers the rest of the firms to behave according to their respective strategies in σ. Taking $F^\sigma_{\mathbf{s}_{-i}}$ as given, let firm i's best response rule for total supply and prices be $\sigma^\sigma_{iz}(\mathbf{s})$ and $\sigma^\sigma_{ip}(\mathbf{s})$, respectively. These functions are such that:*

1. *$\sigma^\sigma_{ip}(\mathbf{s}) = \bar{p}^\sigma_i(\sigma_{iz}(\mathbf{s}); \mathbf{s}_{-i})$, where $\bar{p}^\sigma_i(z_i; \mathbf{s}_{-i})$ is continuous and strictly decreasing in z_i; and*

2. $\sigma_{iz}^{\sigma}(\mathbf{s})$ *has the following form:*

$$\sigma_{iz}^{\sigma}(\mathbf{s}) = \begin{cases} s_i^{*\sigma}(\mathbf{s}_{-i}) & \textit{if } s_{it} \leq \underline{s}_i^{\sigma}(\mathbf{s}_{-i}) \\ s_{it} & \textit{if } s_{it} > \underline{s}_i^{\sigma}(\mathbf{s}_{-i}), \end{cases} \tag{10}$$

where $s_i^{*\sigma}$ and $\underline{s}_i^{\sigma}$ are scalars, with $s_i^{*\sigma} > \underline{s}_i^{\sigma} \ \forall \mathbf{s}_{-i}$, and the following definitions:

$$s_i^{*\sigma}(\mathbf{s}_{-i}) \equiv \arg \max_{\{z_i\}} Q_i^{\sigma}(z_i, \bar{p}_i(z_i); \mathbf{s}_{-i}), \tag{11}$$

$$\underline{s}_i^{\sigma}(\mathbf{s}_{-i}) \equiv \inf\{s_i | Q_i^{\sigma}(s_i^{*\sigma}, \bar{p}_i(s_i^{*\sigma}); \mathbf{s}_{-i}) - k \leq Q_i^{\sigma}(s_i, \bar{p}(s_i); \mathbf{s}_{-i})\}. \tag{12}$$

Proof: See Appendix A.3.

The proposition shows that consideration of oligopolistic competition does not affect the optimality of (S, s) inventory rules.[8] Fixed ordering costs generate infrequent orders. The upper band $s_i^{*\sigma}$ is defined as the optimal order quantity when the firm has no inventory on hand, i.e., the optimal inventory level. The lower band $\underline{s}_i^{\sigma}$ is the smallest value of inventory such that the desired order quantity is zero. This order policy might appear to be a very natural and intuitive strategy. However, as shown in the appendix the value function is not concave such that a much more complex decision rule could in principle be optimal. Additionally, oligopolistic competition assures that no additional assumption on prices like the "no expected loss condition" of Hall and Rust (2007) is necessary for the optimal trading strategy to be of the (S, s) form.[9]

This (S, s) inventory behavior together with demand uncertainty generates cyclical patterns in prices. The optimal price is a strictly decreasing function of a firm's inventory on hand z_i as the positive probability of stockouts creates strategic substitutability between prices and inventories. Thus, the price increases between two orders when the stock level decreases and it drops down when new orders are placed. This is the case as with low inventories the optimal expected demand is lower and hence the optimal price is higher. When the level of inventories decreases between two orders, the probability of stockout increases and so expected sales decrease and become more inelastic with respect to the price. Thus, the optimal price increases between two orders, and decreases when the elasticity of sales goes up as the result of positive orders.

The largest price increase occurs just after a positive order and the increments tend to be smaller when we approach to the next positive order. The reason for this behavior is that the cyclical path of prices generates a cyclical behavior in sales. The largest sales and, consequently, the largest stock reductions and price increases, occur just after a positive order.

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[8]However, as thresholds depend on the competitors' inventories, we have an $(S(\mathbf{s}_{-i}), s(\mathbf{s}_{-i}))$ decision rule.

[9]The "no expected loss condition" requires that the exogenous nonconstant retail price exceeds a certain (endogenous) nonconstant threshold any time. With endogenous prices, we do not need to impose such a condition.

The interesting result here is that the pricing behavior in this model can generate cross-sectional price dispersion with cyclical patterns even without menu cost. The magnitude of this price dispersion will depend on the magnitude of lump-sum ordering costs, the sensitivity of the price elasticity of sales to changes in the probability of stockout, and the degree of correlation between the demand shocks at individual firms.

4   Conclusion

We have shown that the best response not only to (S, s) strategies but to any strategy is an (S, s) rule. This result extends earlier findings of models without price competition (Hall & Rust, 2007) and models without strategic competition (Aguirregabiria, 1999) where fixed ordering costs generate infrequent orders. Thus, the (S, s) policy might appear to be a very robust strategy. However, it is not hard to change assumptions in ways that destroy its optimality.

Additionally, with strategic competition in prices (S, s) inventory behavior together with demand uncertainty generates cyclical pattern in prices.

The model developed in this paper provides a very promising alternative for studying commodity markets.

Commodity prices are extremely volatile and papers of the respective literature strand are concerned whether theory is capable of explaining the actual behavior of prices. The more recent literature on this topic (see for example Deaton & Laroque, 1992, 1996, and Pindyck, 1994) builds on the supply and demand tradition (see e.g. Ghosh, Gilbert, & Hughes Hallett, 1987, for a review), but with explicit modeling of the behavior of competitive speculators who hold inventories of commodities in the expectation of making profits.[10] However, perfect competition and the absence of lump-sum ordering cost is always assumed in these papers. The studies are trying to explain extremely volatile prices as a result of exogenous shocks by modeling the behavior of competitive speculators holding inventories.

Results are rather unsatisfying: In contrast to the models' predictions, real price fluctuations are not randomly distributed over time and this autocorrelation cannot be explained by these types of models. In addition, some probably important characteristics of commodity markets are not captured in this literature. Studies of these characteristics (e.g. Carter & MacLaren, 1997, and Slade & Thille, 2006) find that commodity markets are best described by oligopoly instead of perfect competition. Besides, lump-sum ordering cost are realistic in some markets (e.g. at London Metal Exchange where orders can result in physical delivery and all contracts assume delivery). Incorporating oligopoly competition and lump sum ordering costs could be important to study the dynamics of some commodity prices. In a model like ours we are able to generate some kind of time dependent pattern which is apparently in line with empirical evidence. This is in contrast

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[10]As even estimating the models is computational demanding authors mostly use simulations.

to the usual hypothesis that price fluctuations are the result of exogenous shocks and therefore randomly distributed over time.

Making use of the developed model it should now be possible to relate findings to commodity price dynamics and show that lump-sum ordering cost and oligopoly competition can be important to explain extremely volatile prices and especially time dependencies in price fluctuations.

However, due to the relatively high complexity of the framework further research requires numerical experiments. By this means, other topics like precise reactions of firms on competitors' orders provide scope for interesting studies. This important work is left for future research.

A   Appendix

A.1   Expected Sales: Proof of Lemma 1

Proof. For notational simplicity, we omit here the firm and time subindexes. By definition, expected sales y^e are:

$$y^e = \int \min\{s + q, d^e \exp(\varepsilon)\}dF(\varepsilon) = d^e \lambda\left(\frac{s + q}{d^e}\right)$$

where $\lambda(x)$ is defined as $\int \min\{x, \exp(\varepsilon)\}dF(\varepsilon)$. The function $\lambda(x)$ has the following properties:

$$\lim_{x \to 0} \lambda(x) = \int \min\{0, \exp(\varepsilon)\})dF(\varepsilon) = 0.$$

Also,

$$\lim_{x \to \infty} \lambda(x) = \lim_{x \to \infty} \int \min\{x, \exp(\varepsilon)\})dF(\varepsilon) = \int \exp(\varepsilon)dF(\varepsilon) = 1.$$

Finally,

$$\lambda'(x) = \int I\{x < \exp(\varepsilon)\}dF(\varepsilon) = 1 - F(\ln x).$$

$\square$

A.2   The "Value" Function: Proof of Lemma 2

Proof. We use backwards induction and first show that the properties of Lemma 2 hold for the finite horizon problem with time horizon equal to T.

Let us consider $Q_{iT}^\sigma(\cdot)$ to represent the profit function in the last period, i.e.,

$$
\begin{aligned}
Q_{iT}^\sigma(z_i, p_i; \mathbf{s}_{-i}) &= -cz_i + p_i y_i^{e\sigma}(z_i, p_i) \\
&= -cz_i + p_i d_i^{e\sigma}(p_i)\lambda\left(\frac{z_i}{d_i^{e\sigma}(p_i)}\right) \\
&= -cz_i + p_i \int \min\left\{z_i; e^{\varepsilon_i} d_i^{e\sigma}(p_i)\right\} dF(\varepsilon_i).
\end{aligned}
$$

Therefore,

$$\frac{\partial Q_{iT}^\sigma(\cdot)}{\partial p_i} = y_i^{e\sigma}(z_i, p_i) + p_i \frac{\partial y_i^{e\sigma}(z_i, p_i)}{\partial p_i},$$

and

$$\frac{\partial^2 Q_{iT}(\cdot)}{\partial p_i^2} = 2\frac{\partial y_i^{e\sigma}(z_i, p_i)}{\partial p_i} + p_i \frac{\partial^2 y_i^{e\sigma}(z_i, p_i)}{\partial p_i^2}. \tag{13}$$

Given that $y_i^{e\sigma}(z_i, p_i) = d_i^{e\sigma}(p_i)\lambda\left(\frac{z_i}{d_i^{e\sigma}(p_i)}\right)$, we have that

$$\frac{\partial y_i^{e\sigma}(z_i, p_i)}{\partial p_i} = \frac{\partial d_i^{e\sigma}(p_i)}{\partial p_i}F(\ln z_i - \ln d_i^{e\sigma}(p_i)),$$

and

$$\frac{\partial^2 y_i^{e\sigma}(z_i, p_i)}{\partial p_i^2} = \frac{\partial^2 d_i^{e\sigma}(p_i)}{\partial p_i^2}F(\ln z_i - \ln d_i^{e\sigma}(p_i)) - \left(\frac{\partial d_i^{e\sigma}(p_i)}{\partial p_i}\right)^2 \frac{f(\ln z_i - \ln d_i^{e\sigma}(p_i))}{d_i^{e\sigma}(p_i)}.$$

Inserting these expressions in equation (13), we get:

$$\frac{\partial^2 Q_{iT}^{\sigma}(\cdot)}{\partial p_i^2} = 2\frac{\partial d_i^{e\sigma}(p_i)}{\partial p_i}F(\ln z_i - \ln d_i^{e\sigma}(p_i))$$

$$+ p_i\left(\frac{\partial^2 d_i^{e\sigma}(p_i)}{\partial p_i^2}F(\ln z_i - \ln d_i^{e\sigma}(p_i))\right.$$

$$\left. - \left(\frac{\partial d_i^{e\sigma}(p_i)}{\partial p_i}\right)^2 \frac{f(\ln z_i - \ln d_i^{e\sigma}(p_i))}{d_i^{e\sigma}(p_i)}\right)$$

$$= F(\ln z_i - \ln d_i^{e\sigma}(p_i))\left(2\frac{\partial d_i^{e\sigma}(p_i)}{\partial p_i} + p_i\left(\frac{\partial^2 d_i^{e\sigma}(p_i)}{\partial p_i^2}\right)\right)$$

$$- \left(\frac{\partial d_i^{e\sigma}(p_i)}{\partial p_i}\right)^2 \frac{f(\ln z_i - \ln d_i^{e\sigma}(p_i))}{d_i^{e\sigma}(p_i)}.$$

The first term is negative because $\left(2\frac{\partial d_i^{e\sigma}(p_i)}{\partial p_i} + p_i\left(\frac{\partial^2 d_i^{e\sigma}(p_i)}{\partial p_i^2}\right)\right)$ is just the second derivative of the function $p_i d_i^{e\sigma}(p_i)$, that is strictly concave by assumption. It is clear that the second term is also negative. Therefore, $\frac{\partial^2 Q_{iT}(\cdot)}{\partial p_i^2} < 0$.

Furthermore, since $\frac{\partial Q_{iT}^{\sigma}(\cdot)}{\partial p_i} = y_i^{e\sigma}(z_i, p_i) + p_i\frac{\partial y_i^{e\sigma}(z_i, p_i)}{\partial p_i}$, we have that

$$\frac{\partial^2 Q_{iT}^{\sigma}(\cdot)}{\partial p_i \partial z_i} = \frac{\partial y_i^{e\sigma}(z_i, p_i)}{\partial p_i} + p_i\frac{\partial^2 y_i^{e\sigma}(z_i, p_i)}{\partial p_i \partial z_i}. \tag{14}$$

As we have shown above, $\frac{\partial y_i^{e\sigma}(z_i, p_i)}{\partial z_i} = \lambda'\left(\frac{z_i}{d_i^{e\sigma}(p_i)}\right) = 1 - F(\ln z_i - \ln d_i^{e\sigma}(p_i))$. We have also shown that $\frac{\partial y_i^{e\sigma}(z_i, p_i)}{\partial p_i} = \frac{\partial d_i^{e\sigma}(p_i)}{\partial p_i}F(\ln z_i - \ln d_i^{e\sigma}(p_i))$, and therefore

$$\frac{\partial^2 y_i^{e\sigma}(z_i, p_i)}{\partial p_i \partial z_i} = \frac{\partial d_i^{e\sigma}(p_i)}{\partial p_i}\frac{f(\ln z_i - \ln d_i^{e\sigma}(p_i))}{z_i}.$$

Inserting these expressions into the equation (14), we get:

$$\frac{\partial^2 Q_{iT}^{\sigma}(\cdot)}{\partial p_i \partial z_i} = 1 - F(\ln z_i - \ln d_i^{e\sigma}(p_i)) + \frac{p_i}{z_i}\frac{\partial d_i^{e\sigma}(p_i)}{\partial p_i}f(\ln z_i - \ln d_i^{e\sigma}(p_i)).$$

With $\eta_d(p_i) \equiv -\frac{\partial d_i^{e\sigma}(p_i)}{\partial p_i}\frac{p_i}{d_i^{e\sigma}(p_i)} > 0$ as the elasticity of expected demand, and $\eta_\lambda\left(\frac{z_i}{d_i^{e\sigma}(p_i)}\right) \equiv -\lambda'\left(\frac{z_i}{d_i^{e\sigma}(p_i)}\right)\frac{z_i}{\lambda(\cdot)d_i^{e\sigma}(p_i)} < 0$ as the elasticity of the $\lambda(\cdot)$-function the above expression can be written as

$$\frac{\partial^2 Q_{iT}^{\sigma}(\cdot)}{\partial p_i \partial z_i} = \lambda'(\cdot)(1 - \eta_d(\cdot)(1 - \eta_\lambda(\cdot))) + \lambda(\cdot)\eta_d(\cdot)\eta_\lambda'(\cdot) \tag{15}$$

with

$$\eta_\lambda'(\cdot) = -\frac{\frac{z_i}{d_i^{e\sigma}(p_i)}\lambda'(\cdot)^2 + \lambda(\cdot)(\lambda'(\cdot) + \frac{z_i}{d_i^{e\sigma}(p_i)}\lambda''(\cdot))}{\lambda(\cdot)^2}.$$

The term $\eta_\lambda'(\cdot)$ is negative as $\lambda'(\cdot) + \frac{z_i}{d_i^{e\sigma}(p_i)}\lambda''(\cdot) = 1 - F(\cdot) - f(\cdot)$ is positive for $1 - F(\cdot) > f(\cdot)$ which is fulfilled by assumption. Thus, the second term of equation (15) is negative.

Now, let's particularize expression (15) at $(z, \bar{p}_T(z))$. We can write

$$\frac{\partial Q_{iT}^{\sigma}(\cdot)}{\partial p_i} = y^{e\sigma}(z_i, p_i)(1 - \eta_d(\cdot)(1 - \eta_\lambda(\cdot)))$$

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such that $1 - \eta_d(\cdot)(1 - \eta_\lambda(\cdot))$ can never be positive at the optimal decision and therefore $\frac{\partial^2 Q_{iT}^\sigma(\cdot)}{\partial p_i \partial z_i} < 0$ holds.

We will now show that if $Q_{it+1}^\sigma(\cdot)$ is strictly concave in prices and prices and supply are strategic substitutes in $t+1$, then $Q_{it}^\sigma(\cdot)$ is strictly concave in prices and prices and supply are strategic substitutes in t as well.

We make use of the fact that the profit function is bounded from above. More specifically,

$$\max_{s_i \geq 0} \max_{\{z_i \geq s_i, p_i \geq 0\}} \left\{ p_i d_i^{e\sigma}(p_i) \lambda \left(\frac{z_i}{d_i^{e\sigma}(p_i)} \right) - c_i z_i - k_i I_{\{z_i > s_i\}} \right\}$$

is smaller than some constant $\tau < \infty$. This property guarantees that for any values of z_i and p_i

$$Q_i^\sigma(z_i, p_i) = \lim_{T \to \infty} Q_{iT}^\sigma(z_i, p_i).$$

Thus, as in $t+1$ the "value" function given as

$$Q_i^\sigma(z_{it+1}, p_{it+1}; \cdot) \equiv -cz_{it+1} + p_{it+1} \int \min \left\{ z_{it+1}; e^{\varepsilon_{it+1}} d_{it+1}^{e\sigma}(p_{it+1}) \right\} dF(\varepsilon_{it+1})$$

$$+ \beta \int V_i^\sigma \left(\max \left\{ 0; z_{it+1} - e^{\varepsilon_{it+1}} d_{it+1}^{e\sigma}(p_{it+1}) \right\}; \cdot \right) dF(\varepsilon_{it+1}) dF_{\mathbf{s}_{-i}}^\sigma(\mathbf{s}_{-it+2} | \mathbf{s}_{-it+1})$$

is strictly concave in prices and prices and supply are strategic substitutes, so is the function in t. This completes the proof. □

A.3   Optimal Decision Rule: Proof of Proposition 1.

Following Scarf (1960), the key to proving that the optimal strategy is of the (S, s) form is to show that the value function V is k-concave. Our proof exploits several properties of k-concave functions.

A real-valued function $f(s)$ is a k-concave function if and only if for every s_0 and s_1 such that $s_0 \leq s_1$ and every scalar $\delta \in (0,1)$:

$$\delta f(s_0) + (1 - \delta) f(s_1) \leq (1 - \delta) k + f(\delta s_0 + (1 - \delta) s_1). \tag{16}$$

Consider the following properties of k−concave functions:

(i) If f is strictly k-concave it has a unique global maximum.

(ii) If f is strictly k-concave, and s^* is the global maximum, then the equation $f(z) = f(s^*) - k$ has two solutions, s^L and s^H with $s^L < s^H$. Furthermore, $f(s) > f(s^*) - k$ if and only if $s \in (s^L, s^H)$.

(iii) If $f(x, y)$ is k-concave in x for any value of y, and k-concave in y for any value of x, and $y^*(x) \equiv \arg\max_y f(x, y)$, then $g(x) \equiv f(x, y^*(x))$ is k-concave.

(iv) If $f_1(\cdot)$ is k_1-concave, $f_2(\cdot)$ is k_2-concave, and α_1, α_2 are two positive scalars, then $\alpha_1 f_1 + \alpha_2 f_2$ is $(\alpha_1 k_1 + \alpha_2 k_2)$-concave.

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Before starting with the formal proof, we will briefly illustrate the main idea of why k-concavity is important.

Consider the k-concave function $V(s)$ to be a firm's value function. If V is a continuous differentiable function from k-concavity $V(s_1) - k - V(s_0) - (s_1 - s_0)V'(s_0) \leq 0$ directly follows. Thus for each local extremum s' with $V'(s') = 0$, it is the case that $V(s') \geq V(s) - k \quad \forall s \geq s'$. This means that each local extremum (minimum) s' is at most k units below a function's maximum right of this local minimum. This property is illustrated in Figure 1. The function on the left hand side is an arbitrary value function that is not k-concave, while the function on the right graph fulfills the condition above.
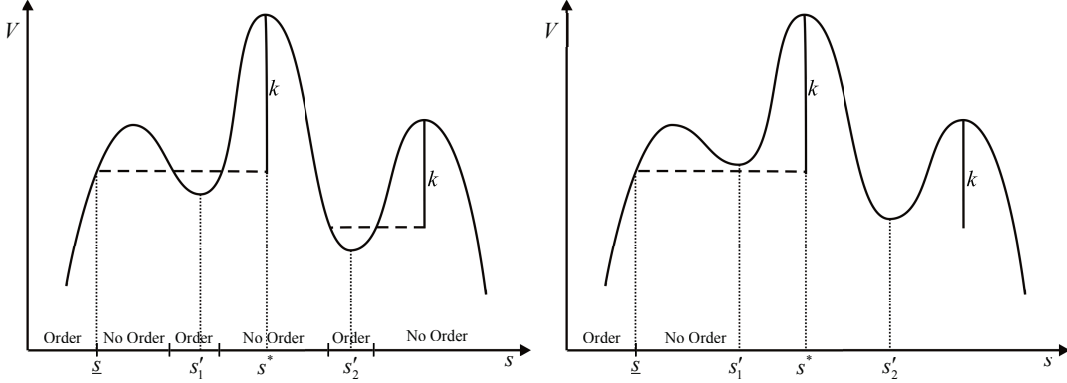

Figure 1: Non-concave value function and respective order decisions when the value function is not k-concave (left) and when it is k-concave (right).

With lump-sum ordering cost of k and a firm's value function like the one depicted on the left hand side a complex optimal order policy results where the firm orders when inventory is below $\underline{s}$ or around s'_1 such that inventory level s^* is attained. Additionally, the firm orders such that an even higher target level is reached when inventory is around s'_2 (which is even above s^*).

With the value function being k-concave like the one depicted on the right hand side, it is easy to see that the optimal strategy is of (S, s) type. In that case firms never order with inventory above s^* and firms never order around a local minimum in between the inventory threshold $\underline{s}$ and the optimal inventory level s^*.

In the following we will make use of this idea with regard to the decision problem of our model.

Proof. Suppose that Q_i^σ is strictly k-concave in z_i for any value of p_i and strictly k-concave in p_i for any value of z_i for all values of $\mathbf{s}_{-it}$.

The optimal price decision can be written as

$$\sigma_{ip}^\sigma(\mathbf{s}) \equiv \bar{p}_i^\sigma(z_i; \mathbf{s}_{-i}).$$

That means, giving the optimal pricing function $\bar{p}_i^\sigma(z; \mathbf{s}_{-i})$ the firm chooses inventory level $\sigma_{iz}^\sigma(\mathbf{s})$ which results in pricing $\sigma_{ip}^\sigma(\mathbf{s})$ as a function of the pre-order inventory.

As $Q_i^\sigma(\cdot)$ is strictly k-concave, $s_i^{*\sigma}(\mathbf{s}_{-i})$ and $\bar{p}_i^\sigma(s_i^{*\sigma}(\mathbf{s}_{-i}), \mathbf{s}_{-i})$ are unique and $\bar{p}_i^\sigma(\cdot, \cdot)$ is a real function. Furthermore, $Q_i^\sigma(z_i, \bar{p}_i^\sigma(z_i); \mathbf{s}_{-i})$ is also strictly k-concave.

By definition of $\sigma_{iz}^\sigma(\mathbf{s})$, $s_i^{*\sigma}(\mathbf{s}_{-i})$, and $\bar{p}_i^\sigma(s_i^{*\sigma}(\cdot), \cdot)$, it is clear that

$$
\sigma_{iz}^\sigma(\mathbf{s}) = \begin{cases} s_i^{*\sigma}(\mathbf{s}_{-i}) & \text{if } Q_i^\sigma(s_i^{*\sigma}, \bar{p}_i^\sigma(s_i^{*\sigma}); \cdot) - k > Q_i^\sigma(s_i, \bar{p}_i^\sigma(s_i); \cdot) \\ s_i & \text{if } Q_i^\sigma(s_i^{*\sigma}, \bar{p}_i^\sigma(s_i^{*\sigma}); \cdot) - k \leq Q_i^\sigma(s_i, \bar{p}_i^\sigma(s_i); \cdot). \end{cases}
$$

Due to the k-concavity of $Q_i^\sigma(z_i, \bar{p}_i^\sigma(z_i); \cdot)$ the equation $Q_i^\sigma(s_i^{*\sigma}, \bar{p}_i^\sigma(s_i^{*\sigma}); \cdot) - k = Q_i^\sigma(s_i, \bar{p}_i^\sigma(s_i); \cdot)$ has only two solutions.

Let these two solutions be $s_i^L(\cdot)$ and $s_i^H(\cdot)$, where $s_i^L(\cdot) \leq s_i^{*\sigma}(\cdot) \leq s_i^H(\cdot)$. Then, k-concavity implies

$$
Q_i^\sigma(s_i^{*\sigma}, \bar{p}_i^\sigma(s_i^{*\sigma}); \cdot) - k \leq Q_i^\sigma(s_i, \bar{p}_i^\sigma(s_i); \cdot) \Leftrightarrow s_i^L(\cdot) \leq s_i(\cdot) \leq s_i^H(\cdot).
$$

It is clear that the conditions $s_i > s_i^H$ and $s_i \leq s_i^H$ do not play any role because the stock level is always lower or equal to $s_i^{*\sigma}$. With $\underline{s}_i^\sigma$ as the smaller of the two solutions by definition we can write the optimal decision as

$$
\sigma_{iz}^\sigma = \begin{cases} s_i^{*\sigma} & \text{if } s_i \leq \underline{s}_i^\sigma, \\ s_i & \text{if } s_i > \underline{s}_i^\sigma. \end{cases}
$$

The according optimal pricing decision for the inventory before ordering is

$$
\sigma_{ip}^\sigma(\mathbf{s}) = \bar{p}^\sigma(\sigma_{iz}(\mathbf{s})) = \begin{cases} \bar{p}^\sigma(s_i^{*\sigma}) & \text{if } s_i \leq \underline{s}_i^\sigma, \\ \bar{p}^\sigma(s_i) & \text{otherwise.} \end{cases}
$$

It further remains to show that Q_i^σ is indeed k-concave.
We proceed in three steps:

(a) If $V_i^\sigma(\mathbf{s})$ is strictly k-concave in s_i, then $Q_i^\sigma(\cdot)$ is strictly k-concave in z_i for any value of p_i.

(b) If $V_i^\sigma(\mathbf{s})$ is strictly k-concave in s_i, then $Q_i^\sigma(\cdot)$ is strictly k-concave in p_i for any value of z_i.

(c) $V_i^\sigma(\mathbf{s})$ is strictly k-concave in s_i.

(a) We will now show that if $V_i^\sigma(\mathbf{s})$ is strictly k-concave in s_i, then $Q_i^\sigma(z_i, p_i; \mathbf{s}_{-i})$ is strictly k-concave in z_i for any value of p_i.

By the first part of the proof, there exist $s_i^{*\sigma}$ and $\underline{s}_i^\sigma$ satisfying $0 \leq \underline{s}_i^\sigma \leq s_i^{*\sigma}$ for which V_i^σ can be represented as

$$V_i^\sigma(\mathbf{s}) = V(Q)_i^\sigma(\mathbf{s}, \sigma_{ip}^\sigma(\mathbf{s})) = \begin{cases} Q_i^\sigma(s_i^{*\sigma}, \bar{p}_i(s_i^{*\sigma}); \mathbf{s}_{-i}) + cs_i - h(s_i) - k & \text{if } s_i \in [0, \underline{s}_i^\sigma), \\ Q_i^\sigma(s_i, \sigma_p(s_i); \mathbf{s}_{-i}) + cs_i - h(s_i) & \text{if } s \geq \underline{s}. \end{cases}$$

$$(17)$$

$V_i^\sigma(\mathbf{s})$ can be extended to be a function defined on $\mathbb{R} \times \mathbb{R}_+^{n-1}$:

$$V_i^\sigma(\mathbf{s}) = \begin{cases} V_i^\sigma(0, \mathbf{s}_{-i}) + cs_i & \text{if } s_i \leq 0, \\ V_i^\sigma(\mathbf{s}) & \text{else,} \end{cases}$$

which is needed as the proof of (c) implies that V_i is k-concave in s_i over $\mathbb{R}$.

We can write Q_i^σ as

$$Q_i^\sigma(\cdot) = Q_i^{\sigma R}(\cdot) + \beta Q_i^{\sigma V}(\cdot),$$

where

$$\begin{aligned} Q_i^{\sigma R}(\cdot) &\equiv -cz_i + p_i \int \min\{z_i; e^{\varepsilon_i} d_i^{e\sigma}(p_i)\} \, dF(\varepsilon_i) \\ &= -cz_i + p_i d_i^{e\sigma}(p_i) \lambda\left(\frac{z_i}{d_i^{e\sigma}(p_i)}\right) \end{aligned}$$

and

$$Q_i^{\sigma V}(\cdot) \equiv \int V_i^\sigma\left(\max\{0; z_{it} - e^{\varepsilon_{it}} d_{it}^{e\sigma}(p_{it})\}; \mathbf{s}_{-it+1}\right) dF(\varepsilon_{it}) dF_{\mathbf{s}_{-i}}^\sigma(\mathbf{s}_{-it+1}|\mathbf{s}_{-it}).$$

Let us now consider the function $\int V_i^\sigma(s_i - e^{\varepsilon_i} d_i^{e\sigma}(\cdot); \cdot) \, dF(\varepsilon_i) dF_{\mathbf{s}_{-i}}^\sigma$. Since each $V_i^\sigma(\cdot)$ is k-concave in s_i over $\mathbb{R}$, and since positive linear combinations of pointwise limits of k-concave functions are k-concave, it follows that $\int V_i^\sigma(s_i - e^{\varepsilon_i} d_i^{e\sigma}(\cdot); \cdot) \, dF(\varepsilon_i) dF_{\mathbf{s}_{-i}}^\sigma$ is k-concave in s_i on $\mathbb{R}$. With $\bar{\varepsilon}_i(\cdot)$ as the value of ε_i for which demand is equal to supply z_i, i.e. $z_i = \exp(\bar{\varepsilon}_i(z_i)) d^{e\sigma}(\cdot)$, we have

$$\begin{aligned} &\int V_i^\sigma(s_i - e^{\varepsilon_i} d_i^{e\sigma}(\cdot); \cdot) \, dF(\varepsilon_i) dF_{\mathbf{s}_{-i}}^\sigma \\ &= \int_{-\infty}^{\bar{\varepsilon}_i(s_i)} V_i^\sigma(s_i - e^{\varepsilon_i} d_i^{e\sigma}(\cdot); \cdot) \, dF(\varepsilon_i) dF_{\mathbf{s}_{-i}}^\sigma \\ &\quad + \int_{\bar{\varepsilon}_i(s_i)}^{\infty} V_i^\sigma(s_i - e^{\varepsilon_i} d_i^{e\sigma}(\cdot); \cdot) \, dF(\varepsilon_i) dF_{\mathbf{s}_{-i}}^\sigma \\ &= \int_{-\infty}^{\bar{\varepsilon}_i(s_i)} V_i^\sigma(s_i - e^{\varepsilon_i} d_i^{e\sigma}(\cdot); \cdot) \, dF(\varepsilon_i) dF_{\mathbf{s}_{-i}}^\sigma + V_i^\sigma(0; \cdot) \int_{\bar{\varepsilon}_i(s_i)}^{\infty} dF(\varepsilon_i) dF_{\mathbf{s}_{-i}}^\sigma \\ &\quad + c \int_{\bar{\varepsilon}_i(s_i)}^{\infty} (s_i - e^{\varepsilon_i} d_i^{e\sigma}(\cdot)) \, dF(\varepsilon_i) \\ &= Q_i^{\sigma V}(s_i, p_i, \cdot) + c \int_{\bar{\varepsilon}_i(s_i)}^{\infty} (s_i - e^{\varepsilon_i} d_i^{e\sigma}(\cdot)) \, dF(\varepsilon_i). \end{aligned}$$

Using the definition of Q_i^σ, we have

$$
\begin{aligned}
Q_i^\sigma(\cdot) =& Q_i^{\sigma R}(\cdot) + \beta Q_i^{\sigma V}(\cdot) \\
=& p_i \int \min\left\{z_i; e^{\varepsilon_i} d_i^{e\sigma}(p_i)\right\} dF(\varepsilon_i) - cz_i \\
& + \beta \int_{-\infty}^{\bar\varepsilon_i(z_i)} V_i^\sigma\left(z_i - e^{\varepsilon_i} d_i^{e\sigma}(\cdot); \cdot\right) dF(\varepsilon_i) dF_{\mathbf{s}_{-i}}^\sigma \\
& + \beta \int_{\bar\varepsilon_i(z_i)}^{\infty} V_i^\sigma\left(z_i - e^{\varepsilon_i} d_i^{e\sigma}(\cdot); \cdot\right) dF(\varepsilon_i) dF_{\mathbf{s}_{-i}}^\sigma \\
& - \beta c \int_{\bar\varepsilon_i(z_i)}^{\infty} \left(z_i - e^{\varepsilon_i} d_i^{e\sigma}(\cdot)\right) dF(\varepsilon_i).
\end{aligned}
$$

The sum of the third and fourth terms in the last equation is k-concave since $\int V_i^\sigma\left(s_i - e^{\varepsilon_i} d_i^{e\sigma}(\cdot); \cdot\right) dF(\varepsilon_i) dF_{\mathbf{s}_{-i}}^\sigma$ is k-concave. Since cz_i is a linear and hence convex function of z_i, a sufficient condition for the k-concavity of $Q_i^\sigma(\cdot)$ is that the function

$$
p_i d_i^{e\sigma}(p_i) \lambda\left(\frac{z_i}{d_i^{e\sigma}(p_i)}\right) - \beta c \int_{\bar\varepsilon_i(z_i)}^{\infty} \left(z_i - e^{\varepsilon_i} d_i^{e\sigma}(\cdot)\right) dF(\varepsilon_i)
$$

is concave in z_i. The function is continuously differentiable in z_i with second derivatives

$$
(p_i - \beta c)(1 - F(\ln z_i - \ln d_i^{e\sigma}(\cdot))).
$$

As $F(\cdot) < 1$, this expression is non-positive and hence Q_i^σ is k-concave as long as $p_i \geq \beta c_i$. (Obviously, a weaker condition for that result exists.)

For proving that Q_i^σ is indeed k-concave we need to show that $\sigma_{ip}^\sigma(\mathbf{s}) - \beta c \geq 0$ holds. Recall

$$
\begin{aligned}
Q_i^\sigma(z_i, p_i; \mathbf{s}_{-i}) \equiv& -cz_i + p_i d_i^{e\sigma}(p_i)\lambda\left(\frac{z_i}{d_i^{e\sigma}(p_i)}\right) \\
& + \beta \int V_i^\sigma\left(\max\left\{0; z_i - e^{\varepsilon_i} d_i^{e\sigma}(p_i)\right\}; \mathbf{s}_{-it+1}\right) dF(\varepsilon_i) dF_{\mathbf{s}_{-i}}^\sigma(\mathbf{s}_{-it+1}|\mathbf{s}_{-i})
\end{aligned}
$$

and

$$
V_i^\sigma(\mathbf{s}_t) = \max_{\{p_i, q_i\}}\left\{Q_i^\sigma(s_{it} + q_i, p_i; \mathbf{s}_{-it}) - (h_i - c_i)s_{it} - k_i I_{\{q_i > 0\}}\right\}.
$$

where the expected sales $d_i^{e\sigma}(p_i)\lambda\left(\frac{z_i}{d_i^{e\sigma}(p_i)}\right)$ are always smaller than or equal to total supply z_i. Let's suppose to the contrary that there is an optimal price $\sigma_{ip}^\sigma < \beta c < c$. In that case $-cz_i + p_i d_i^{e\sigma}(p_i)\lambda\left(\frac{z_i}{d_i^{e\sigma}(p_i)}\right)$ would be negative. Thus, without a new order the current value $V_i^\sigma(\mathbf{s}_t)$ would be smaller than the expected value $V_i^\sigma(\mathbf{s}_{t+1})$ after selling the goods at price $\sigma_p(s_t)$ although the inventory is larger, i.e., $s_{it} > s_{it+1}$. This cannot be the case in equilibrium. The same is true in the case with ordering. Ordering goods and simultaneously selling them for a price lower than the purchase price cannot be an optimal strategy. Thus, the optimal price σ_{ip}^σ is always greater c.

(b) We will show that if $V_i^\sigma(\mathbf{s})$ is strictly k-concave in s_i, then $Q_i^\sigma(\cdot)$ is strictly k-concave in p_i for any value of z_i.

We can represent the function $Q_i^{\sigma R}(\cdot)$ as $-cz_i + p_i y^{\sigma e}(z_i, p_i; \mathbf{s}_{-i})$, where $y^{\sigma e}(\cdot)$ is the expected sales function. The function $Q_i^{\sigma R}(\cdot)$ is the same as the function Q_i^σ at the last period Q_{iT}^σ. We have shown in the proof of Lemma 2 that this function is convex.

Therefore, $\frac{\partial^2 Q_i^{\sigma R}(\cdot)}{\partial p_i^2} < 0$.

An argumentation analogous to part (a) yields a similar sufficient condition for the k-concavity of $Q_i^\sigma(\cdot)$ in p_i, namely that the function

$$p_i d_i^{e\sigma}(p_i)\lambda\left(\frac{z_i}{d_i^{e\sigma}(p_i)}\right) - \beta c \int_{\bar{\varepsilon}_i(z_i)}^{\infty} (z_i - e^{\varepsilon_i} d_i^{e\sigma}(\cdot))\, dF(\varepsilon_i)$$

is concave in p_i. The function is continuously differentiable in p_i with a second derivative that is negative. Therefore, $Q_i^\sigma(\cdot)$ is k-concave in p_i.

(c) Finally, we show that $V_i^\sigma(\mathbf{s})$ is strictly k-concave in s_i.

Like in proof of Lemma 2 we make use of the fact that the profit function is bounded from above. This property guarantees that for any value of s_i

$$V_i^\sigma(s_i; \cdot) = \lim_{T \to \infty} V_{iT}^\sigma(s_i; \cdot)$$

with $V_{iT}^\sigma(s_i)$ as the value function for the finite horizon problem with time horizon equal to T. We prove k-concavity by induction.

For $T = 1$ we have $Q_{i1}^\sigma(\cdot)$ is strictly concave in z_i and p_i due to (a) and (b). Using the result of the first part of the proof, the optimal decision for this one-period problem has the form of equations (9) and (10). Hence, the value function of this one period problem is

$$V_{i1}^\sigma(s_i, \cdot) = I(s_i < \underline{s}_{i1}^\sigma)\left(Q_{i1}^\sigma(s_{i1}^{*\sigma}, \bar{p}_{i1}^\sigma(s_{i1}^{*\sigma})) - k\right) + I(s_i \geq \underline{s}_{i1}^\sigma)Q_{i1}^\sigma(s_i, \bar{p}_{i1}^\sigma(s_i, \cdot)) - (h_i - c_i)s_i.$$

With $Q_{i1}^\sigma(\cdot)$ being concave, it is simple to verify that $V_{i1}^\sigma(s_i, \cdot)$ fulfills the definition of strict k-concavity.

Assume now that for any $t \geq 1$, $V_{it}^\sigma(s_i, \cdot)$ is strictly k-concave. Then,

$$Q_{it+1}^\sigma(z_i, \bar{p}_{it+1}^\sigma(\cdot); \mathbf{s}_{-it+1}) = -cz_i + \bar{p}_{it+1}^\sigma(\cdot)d_i^{e\sigma}(\bar{p}_{it+1}^\sigma(\cdot))\lambda\left(\frac{z_i}{d_i^{e\sigma}(\bar{p}_{it+1}^\sigma(\cdot))}\right)$$

$$+ \beta \int V_i^\sigma\left(\max\left\{0; z_i - e^{\varepsilon_i}d_i^{e\sigma}(\bar{p}_{it+1}^\sigma(\cdot))\right\}; \cdot\right)dF(\varepsilon_i)dF_{\mathbf{s}_{-i}}^\sigma(\mathbf{s}_{-it+2}|\mathbf{s}_{-it+1}).$$

As $\bar{p}_{it+1}^\sigma(\cdot)d_i^{e\sigma}(\bar{p}_{it+1}^\sigma(\cdot))\lambda\left(\frac{z_i}{d_i^{e\sigma}(\bar{p}_{it+1}^\sigma(\cdot))}\right) - cz_i$ is again strictly concave and $V_{it}^\sigma(s_i, \cdot)$ is strictly k-concave, due to property (iv) of k-concave functions, $Q_{it+1}^\sigma(z_i, \bar{p}_{it+1}^\sigma(\cdot); \cdot)$ is also strictly k-concave. Hence, the optimal decision has again the form of equations (9) and (10) and the value function of this finite-horizon problem is

$$V_{it+1}^\sigma(s_i, \cdot) = I(s_i < \underline{s}_{it+1}^\sigma)\left(Q_{it+1}^\sigma(s_{it+1}^{*\sigma}, \bar{p}_{it+1}^\sigma(s_{it+1}^{*\sigma})) - k\right)$$

$$+ I(s_i \geq \underline{s}_{it+1}^\sigma)Q_{it+1}^\sigma(s_i, \bar{p}_{it+1}^\sigma(s_i, \cdot)) - (h_i - c_i)s_i.$$

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Similar to $V_{i1}^\sigma(s_i, \cdot)$, this value function is strictly k-concave which completes the proof by induction. Therefore, $V_i^\sigma(s_i; \cdot) = \lim_{T\to\infty} V_{iT}^\sigma(s_i; \cdot)$ is strictly k-concave.

This completes the proof of the optimality of the described ordering strategy.

Properties of the optimal price.   We complete the proof of Proposition 1 by showing that $\bar{p}(\cdot)$ is a continuous and strictly decreasing function.

The function $\bar{p}_i^\sigma$ is the value of p_i that maximizes Q_i^σ in p_i for a given z_i. Since Q_i^σ is continuously differentiable and strictly concave in prices, $\bar{p}_i^\sigma(z; \mathbf{s}_{-i})$ is implicitly defined by the first order condition $\frac{\partial Q_i^\sigma(z_i, \bar{p}; \mathbf{s}_{-i})}{\partial p_i} = 0$. By the implicit function theorem, we have that $\frac{d\bar{p}_i(z_i)^\sigma}{dz_i} = -\frac{\partial^2 Q_i^\sigma(z_i, \bar{p}_i)/\partial p_i \partial z_i}{\partial^2 Q_i^\sigma(z_i, \bar{p}_i)/\partial p_i \partial p_i}$, that by Lemma 2 is negative.

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

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