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# Is Rational Speculation in the Presence of Positive Feedback Traders Destabilizing?

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## **Abstract**

DeLong (1990a) et al. show that in the presence of positive feedback traders rational speculation can be destabilizing, in that it drives the price of a risky asset above its expected value. A generalization of their seminal model with additional trading dates and an additional informative signal yields further interesting insights: it helps clarify when prices overreact, underreact, or even move in the “wrong” direction; when rational speculation is destabilizing or stabilizing; and whether overreaction is a symptom of market inefficiency or a manifestation of informational efficiency.

JEL classification: G12, G14

Key words: market efficiency, positive feedback trading

# 1 Introduction

In a pioneering article, De Long et al. (1990a, henceforth: DSSW) present a finite-horizon model of an asset market with noise traders in which rational agents' utility-maximizing trading strategies destabilize, rather than stabilize, the asset price. In their model, noise traders are engaged in positive feedback trading. After positive news about the asset payoff, they take a long position at the trading date before the payoff is realized. Since the asset is in zero net supply, equality of supply and demand implies that rational traders go short. Due to fundamental risk, the payoff is still uncertain from the final trading date's point of view, so the asset's price must exceed the expected payoff in order to compensate rational traders for taking the risky short position. In the absence of rational traders, by contrast, price equals expected value, conditional on current information, at all trading dates. In this sense, rational speculation destabilizes price.

We consider a generalized version of the DSSW model (with a noiseless signal) that incorporates additional trading dates and an additional informative signal. For one thing, this allows us to check how the price response depends on the timing of the arrival of news and of noise traders' feedback trades. For another, we can check how the price response to earlier news differs from the price reaction to the final piece of news. This yields several interesting results:

**Result 1:** Rational speculation can be stabilizing: the price response to news can be less pronounced with than without rational traders.

**Result 2:** If there is only one signal, then only *one* parameter from the noise traders' feedback trading strategy affects prices, viz., the feedback parameter which determines the impact of the price rise at the date when the signal arrives on asset demand at the final trading date. If noise trader demand reacts less strongly to price rises further back in the past, then the degree of overreaction is low if the signal arrives long before the asset pays off.

**Result 3:** Following Loewenstein and Willard (2006, henceforth LW), one can interpret the risk created by noise traders as fundamental risk and the resulting price sequence as reflecting this fundamental risk. This view receives support from the fact that in the absence of rational speculators, as the number of trading dates grows, the price slowly approaches the level that is obtained at once in the presence of rational traders. On this interpretation, the market with rational speculators is efficient, and no overreaction without rational speculators merely reflects that the time horizon is too short for convergence to fundamental value to take place.

**Result 4:** If there is positive feedback only to the most recent price rise, then the asset price displays damped fluctuations in the absence of rational speculators.

**Result 5:** If there are two signals, the price response to the second signal is essentially the same as the price response in the model with only one signal.

**Result 6:** In response to the first signal, the asset price may underreact at first. If so, then it overreacts later on, however. In this sense, DSSW's overreaction result remains valid in the generalized model.

**Result 7:** It may happen that in response to a positive first signal rational speculators short the asset and the asset price falls. This is because the returns on investment before the arrival of the second signal and at the final trading date covary positively, so that a short position at the former date provides a hedge against low returns on the final-date investment position.

**Result 8:** The price response to the first signal can be characterized by mean reversion: in response to good news, the price rises first and then falls, possibly even below the asset's expected payoff.

**Result 9:** The price reaction to the first signal tends to be moderate both for low and for high relative proportions of rational versus passive investors. The strength of the price reaction increases dramatically as the proportion approaches an intermediate value.

Shiller (1981, 2005) and many others point out that the observed degree of stock prices volatility is hard to reconcile with fundamental valuation. De Bondt and Thaler (1985), Jegadeesh and Titman (1993), and Dimson et al. (2008), among many others, argue that stock prices show short-term momentum and long-run mean reversion, so buying short-term winners and long-term losers and shorting short-term losers and long-term winners yields excess returns, apparently contradicting even weak-form market efficiency. This paper contributes to the literature which attempts to provide a theoretical explanation for high volatility, momentum, and mean reversion based on the interaction of noise traders and rational investors. The seminal papers in this strand of the literature (DSSW, De Long et al., 1990b, and Shleifer and Vishny, 1997) are still heavily cited by empirical, experimental, and behavioral studies.<sup>1</sup> But relatively little progress has been made in generalizing and extending the original models (see LW and Arnold, 2009).

Section 2 describes the model. Section 3 derives the solution to rational speculators' utility maximization problem. Section 4 considers the special case of the model with only one signal, which covers the DSSW model and an example with stabilizing rational speculation as special cases. The general case is treated in Section 5. Section 6 concludes.

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<sup>1</sup>Google Scholar counts more than 1,500 citations of the DSSW paper as of early 2012.

## 2 Model

There are  $T + 2$  dates ( $t = 0, 1, \dots, T + 1$ ,  $T \geq 2$ ) and three types of agents: a set of measure one of positive feedback traders, a set of measure  $\mu$  of informed rational speculators, and a set of measure  $1 - \mu$  of passive investors, where  $0 \leq \mu \leq 1$ . Superscripts  $f$ ,  $r$ , and  $e$  are used in order to distinguish variables referring to positive feedback traders, informed speculators, and passive investors, respectively.<sup>2</sup>

There are one consumption good and one safe and one risky asset. Assets pay off in terms of the consumption good. The safe asset is in perfectly elastic supply. Its rate of return is zero. The supply  $S$  ( $\geq 0$ ) of the risky asset is exogenous. The asset pays  $v + \Phi + \theta$  at  $T + 1$  (and no dividends before  $T + 1$ ), where  $v$  ( $\geq 0$ ) is safe and  $\Phi$  and  $\theta$  are random variables.  $\theta$  has mean zero and a positive, finite variance  $\sigma_\theta^2$ .  $\Phi$  is the sum of two i.i.d. shocks  $\phi'$  and  $\phi''$ , which both have mean zero, a finite variance, and symmetric density. We allow for  $\sigma_{\phi'}^2 = 0$ , so that there is only one signal. We restrict attention to the case of noiseless signals: rational speculators observe the realizations of  $\phi'$  and  $\phi''$  at  $t^{r'}$  and  $t^{r''}$ , respectively ( $1 \leq t^{r'} < t^{r''} \leq T$ ). Passive investors learn about the realizations at  $t^{e'}$  and  $t^{e''}$ , respectively ( $t^{r'} < t^{e'}$ ,  $t^{r''} < t^{e''} \leq T$ ). No-one receives a signal about  $\theta$  before  $T + 1$ . If the asset is interpreted as a proxy for the stock market,  $T + 1$  should be loosely interpreted as a date at which the profitability of the firms and their subsequent dividends change markedly due to changes in firms' internal organization or the market environment and  $t^{r'}$  and  $t^{r''}$  as dates when information about the firms' probable future success leaks out.

Agents consume only at the final date  $T + 1$ . Let  $p_t$  denote the price of the asset and  $D_t^i$  ( $i = f, r, e$ ) the amount of the asset held by a type- $i$  trader at date  $t$  ( $= 0, 1, \dots, T$ ). The feedback traders' demand for the asset is

$$D_0^f = D_1^f = S, \quad D_t^f = S + \sum_{l=1}^{t-1} \beta_l \Delta p_{t-l}, \quad t = 2, 3, \dots, T, \quad (1)$$

where  $\Delta p_t = p_t - p_{t-1}$  and  $\beta_l \geq 0$  ( $l = 1, 2, \dots, T - 1$ ). That is, feedback traders absorb the supply of the asset if the price is constant. After a price increase (decrease) their demand is higher (lower).

The passive investors' demand is

$$D_t^e = \alpha(v + E_t^e \Phi - p_t), \quad t = 0, 1, \dots, T. \quad (2)$$

$E_t^e \Phi$  is the expectation of  $\Phi$  ( $= \phi' + \phi''$ ) based on the signals passive investors observe up to date  $t$ , i.e.,  $E_t^e \Phi = 0$  for  $t < t^{e'}$ ,  $E_t^e \Phi = \phi'$  for  $t^{e'} \leq t < t^{e''}$ , and  $E_t^e \Phi = \Phi$  for  $t^{e''} \leq t$ . That is, passive

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<sup>2</sup>It is understood that there is a well-defined probability space, random variables are measurable functions from the sample space to the reals, and expected value is defined as Lebesgue integral.

investors buy if they perceive that the dividend exceeds the asset price, and vice versa. They do not use the current price level to update their expectation of  $\Phi$ .<sup>3</sup> Following DSSW (p. 386), we assume that  $\alpha = 1/(2\gamma\sigma_\theta^2) > \beta_l$  for all  $l$ . This assumption ensures that the demand functions of rational speculators and passive investors are identical, once all signals have arrived. Informed rational speculators are the only utility maximizing agents in the model. Their preferences are represented by the mean-variance utility function  $\mu - \gamma\sigma^2$  ( $\gamma > 0$ ), where  $\mu$  and  $\sigma^2$  are the mean and the variance of their final wealth, respectively.

**Definition:** Prices  $p_t$  and demands  $D_t^f$ ,  $D_t^r$ , and  $D_t^e$  ( $t = 0, 1, \dots, T$ ) are an equilibrium if for all  $t = 0, 1, \dots, T$ ,  $D_t^f$  satisfies (1),  $D_t^e$  satisfies (2),  $D_t^r$  is the time-consistent solution to the mean-variance utility maximization problem, given current information, and the market for the risky asset clears:

$$D_t^f + \mu D_t^r + (1 - \mu)D_t^e = S. \quad (3)$$

We confine attention to equilibria in which the asset price is a linear function of the realized shocks with intercept  $v$  when  $\mu > 0$ :

$$p_t = \begin{cases} v; & 0 \leq t < t^{r'} \\ v + (1 + \nu')\phi'; & t^{r'} \leq t < t^{r''} \\ v + (1 + \lambda')\phi' + (1 + \lambda'')\phi''; & t^{r''} \leq t \leq T \end{cases} \quad (4)$$

for some  $\nu'$ ,  $\lambda'$ , and  $\lambda''$  yet to be determined. We say that the price overreacts to the date- $t^{r'}$  signal at  $t^{r'}$  (at  $t^{r''}$ ) if  $\nu' > 0$  (if  $\lambda' > 0$ ) and that it overreacts to the date- $t^{r''}$  signal if  $\lambda'' > 0$ . This definition does not relate to a notion of “fundamentally justified” price reaction, an issue we discuss below. From (4), the asset price satisfies the martingale property that the price reaction to contemporary shocks is unpredictable. However, for the rational traders, who observe  $\phi'$  at  $t^{r'}$  ( $< t^{r''}$ ), the date- $t^{r''}$  price change

$$p_{t^{r''}} - p_{t^{r''}-1} = (\lambda' - \nu')\phi' + (1 + \lambda'')\phi'' \quad (5)$$

contains a predictable portion  $(\lambda' - \nu')\phi'$ .

The DSSW model is the special case with  $T = 2$ ,  $\sigma_{\phi''} = 0$  (so that  $\Phi = \phi'$ ),  $t^{r'} = 1$ ,  $t^{e'} = 2$ ,  $v = 0$ ,  $S = 0$ ,  $\Phi \in \{-\phi, 0, \phi\}$  ( $\phi > 0$ ), and  $\theta$  distributed normally. Put the other way round, our model is the DSSW model with a noiseless signal augmented to include additional trading dates and an additional informative signal.

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<sup>3</sup>If they used the current price to update their beliefs, they would behave just like the rational speculators. So this variant of the model is analogous to the special case with  $\mu = 1$ .

### 3 Investment behavior

This section deals with the rational speculators' dynamic mean-variance portfolio selection problem.<sup>4</sup> Let  $W_t^r$  denote a rational speculator's wealth at date  $t$  ( $= 0, 1, \dots, T + 1$ ). Initial wealth  $W_0^r$  ( $> 0$ ) is exogenous. Final wealth is

$$W_{T+1}^r = W_t^r + \sum_{t'=t}^{T-1} (p_{t'+1} - p_{t'})D_{t'}^r + (v + \Phi + \theta - p_T)D_T^r, \quad t = 0, 1, \dots, T - 1. \quad (6)$$

The second term on the right-hand side is the sum of the capital gains at dates  $t + 1$  through  $T$ . The final term on the right-hand side is the return on the date- $T$  asset holdings.

Solutions to the investors' mean-variance portfolio selection problem are found recursively. At the final trading date, the marginal increase in expected final wealth due to an increase in  $D_T^r$  is  $v + \Phi - p_T$ . The conditional variance of final wealth is  $\sigma_\theta^2 D_T^r{}^2$ , so the marginal cost of an increase in  $D_T^r$  in terms of additional volatility is  $(2\gamma\sigma_\theta^2 D_T^r) = D_T^r/\alpha$ . Mean variance optimization yields

$$D_T^r = \alpha(v + \Phi - p_T). \quad (7)$$

As emphasized by DSSW (pp. 385-6), the presence of fundamental dividend risk implies that there is not an arbitrage opportunity so the date- $T$  demand is bounded. From (4), the asset price is constant except when new information arrives, i.e.,  $p_t = p_{t+1}$  for  $t < T$  except  $t = t^{r'} - 1$  (if  $\sigma_\phi^2 > 0$ ) and  $t = t^{r''} - 1$ . From (6),  $D_t^r$  is indeterminate then, since it does not affect final wealth  $W_{T+1}^r$ . At  $t^{r''} - 1$  and  $t^{r'} - 1$  (if  $\sigma_\phi^2 > 0$ ), the asset price is risky, and rational speculators maximize their mean-variance utility, given their optimum demands at subsequent dates:

**Proposition 1:** *A rational trader's demands at  $t^{r''} - 1$  and  $t^{r'} - 1$  are given by*

$$D_{t^{r''}-1}^r = \frac{v + (1 + \lambda')\phi' - p_{t^{r''}-1}}{2\gamma(1 + \lambda'')^2\sigma_{\phi''}^2} - \frac{2\alpha\lambda'\lambda''\phi'}{1 + \lambda''} \quad (8)$$

and, if  $\sigma_\phi^2 > 0$ ,

$$D_{t^{r'}-1}^r = \frac{v - p_{t^{r'}-1}}{2\gamma(1 + \nu')^2\sigma_{\phi'}^2}.$$

**Proof:** The proof is straightforward but tedious (see the Appendix).

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<sup>4</sup>Finding the time-consistent solution to the rational speculators' dynamic mean-variance optimization problem is the major obstacle to further generalization, as there is no "cookbook recipe" for solving such problems (see Li and Ng, 2000, and Steinbach, 2001, Section 2). Anyway, as we will see in Section 5, the version of the model with two signals is sufficiently rich to create virtually any conceivable pattern of asset price movements.

The first fraction in (8) is similar to the final-date demand function  $D_T^r$ : it is the expected date- $t^{r''}$  price change divided by  $2\gamma$  times the conditional price variance. To interpret the second term, suppose  $\lambda' > 0$  and  $\lambda'' > 0$  and consider  $\phi' > 0$ . For  $\phi'' (> -\lambda'\phi'/\lambda'')$  not “too small”, the asset price exceeds expected payoff by  $\lambda'\phi' + \lambda''\phi''$ , so that rational speculators short ( $-D_T^r =$ )  $\alpha(\lambda'\phi' + \lambda''\phi'')$  ( $> 0$ ) assets. An increase in  $\phi''$  then raises both (the date- $T$  price and, therefore) the profitability and the size of the short sale. Thus, increases in  $\phi''$  change the investment opportunities at the final trading date in a favorable way. From (5), increases in  $\phi''$  also raise the one-period holding return on investments at  $t^{r''} - 1$ . Hence, there is positive correlation between the returns on investment at  $t^{r''} - 1$  and at  $T$ , and raising the period- $t^{r''} - 1$  investment  $D_{t^{r''}-1}^r$  increases the covariance of returns. The second term in (8) represents the negative impact of this covariance effect on asset demand at  $t^{r''} - 1$ . Due to the covariance effect, rational traders may take a short position after a positive first signal, even though the expected price change (the numerator of the first fraction) is positive. At date  $t^{r'}$ , by contrast, given the maintained independence and symmetry assumptions, asset demand *is* proportional to the expected price change. If there is only one signal, the covariance term also drops out of (8).

## 4 One signal

### 4.1 The DSSW model

This subsection reviews the DSSW model.<sup>5</sup> Following DSSW (p. 388), we analyze the impact of rational speculation on equilibrium prices by comparing prices with and without rational informed traders (i.e., with  $\mu > 0$  and  $\mu = 0$ ).

**Proposition 2:** *Let  $T = 2$ ,  $\sigma_{\phi'} = 0$  (so that  $\Phi = \phi''$ ),  $t^{r''} = 1$ , and  $t^{e''} = 2$ . Then the equilibrium prices are  $p_0 = v$  and*

$$p_1 = p_2 = v + \left(1 + \frac{\beta_1}{\alpha - \beta_1}\right) \Phi \quad (9)$$

for  $\mu > 0$  and

$$p_1 = v, \quad p_2 = v + \Phi$$

for  $\mu = 0$ .<sup>6</sup>

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<sup>5</sup>As laid out in Section 2, we allow for  $v \geq 0$ ,  $S \geq 0$ , and  $\theta$  non-normal. This minor generalization is inessential and straightforward to handle.

<sup>6</sup>To ensure non-negativity of the asset's payoff and its price, one has to assume  $\underline{\phi}'' + \underline{\theta} \geq -v$  and  $\underline{\phi}'' \geq -(1 - \beta_1/\alpha)v$ , where  $\underline{\phi}''$  and  $\underline{\theta}$  denote the lower bounds of the supports of  $\phi''$  and  $\theta$ , respectively. Similar restrictions ensure non-negativity of payoffs and prices in the versions of the model considered subsequently.



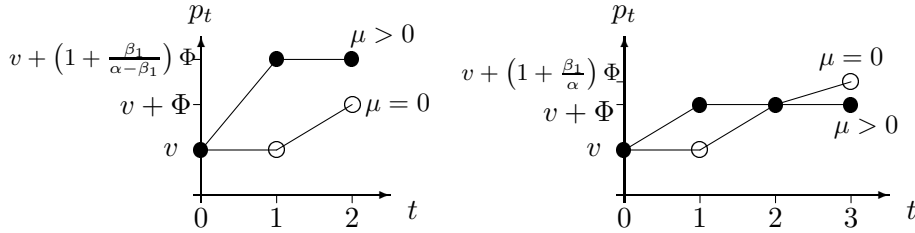


Figure 1: Impulse response to  $\Phi > 0$ : destabilizing (left panel) or stabilizing (right panel) rational speculation

**Proof:** The demands at the final trading date are  $D_2^f = S + \beta_1(p_1 - p_0)$  and  $D_2^e = D_2^r = \alpha(v + \Phi - p_2)$ . From (4), we have  $p_0 = v$  and  $p_1 = p_2$  for  $\mu > 0$ . So the market clearing condition (3) for date 2 can be written as  $0 = \beta_1(p_2 - v) + \alpha(v + \Phi - p_2)$ . Solving for  $p_2$  yields the expression in (9). From (1), (2), and Proposition 1 with  $t^{r''} = 1$ , it follows that  $p_0 = v$  (cf. (4)) is in fact an equilibrium price.<sup>7</sup>

For  $\mu = 0$ ,  $D_t^e = \alpha(v - p_t)$  for  $t = 0, 1$ . Market clearing yields  $p_t = v$ . This implies  $D_2^f = S$ . Together with  $D_2^e = \alpha(v + \Phi - p_2)$ , market clearing yields  $p_2 = v + \Phi$ . ||

As emphasized by DSSW (p. 388), rational speculation is destabilizing, in that the price reaction to news is stronger with than without rational traders (see the left panel of Figure 1).

An important, albeit semantic, question is whether price equals fundamental value or there is a bubble. LW (Section II) argue forcefully that a deviation of the equilibrium price sequence in a model with noise traders from the price sequence which prevails in the absence of noise traders should not automatically be identified as a bubble, since noise trading causes consumption (i.e., fundamental) risk.<sup>8</sup> DSSW say that “the price is . . . away from fundamentals” (p. 388), so there is a “bubble” (p. 392), even if not a “rational bubble” as, for instance, in Tirole (1982). However, following LW, one might object that the equilibrium returns compensate rational traders for the consumption risk they bear as a result of the non-rational agents’ trading strategies. From (2)

<sup>7</sup>DSSW (p. 387) treat date 0 as a “reference period” with no trade, at which “the price is set at its fundamental value”. Our generalized analysis shows that  $p_0 = v$  in equilibrium if date 0 is a trading date like the subsequent ones.

<sup>8</sup>Their analysis does not apply directly here. This is because the (generalized) DSSW model is not a “DSSW-style model” in the sense of their Definition 2 (LW, p. 240, which refers to DeLong et al., 1990b), since aggregate consumption and the asset price are not perfectly correlated (cf. LW, Proposition 1, p. 241). To the contrary, aggregate consumption is zero at all trading dates and, therefore, uncorrelated with the asset price. Put differently, there is an arbitrage opportunity in DeLong et al. (1990b) if there is a lower bound on the price. By contrast, the equilibrium prices in Proposition 2 (and subsequent versions of the generalized model) are arbitrage-free, even though in general mean-variance optimization does not necessarily exploit arbitrage opportunities.

and (7), the price level that makes a rational investor (and a passive investor for that matter) willing to take the investment position  $D_2^r$  at date 2 is  $p_2 = v + \Phi - D_2^r/\alpha$ . The asset price is equal to the expected payoff if, and only if,  $D_2^r = 0$ . Whenever rational speculators take a non-zero investment position, they must be compensated for the risk they take. The positive feedback traders inelastically demand  $D_2^f = S + \beta_1[1 + \beta_1/(\alpha - \beta_1)]\Phi$  units of the asset at date 2. The equilibrium price  $p_2$  in (9) includes a premium which makes the rational traders willing to take the short position  $-D_2^r = D_2^f - S$  necessary for market clearing after a positive signal ( $\Phi > 0$ ):

$$v + \left(1 + \frac{\beta_1}{\alpha - \beta_1}\right) \Phi = v + \Phi + \frac{1}{\alpha}\beta_1 \left(1 + \frac{\beta_1}{\alpha - \beta_1}\right) \Phi.$$

Conversely, a discount  $v + \Phi - p_2 = -\beta_1\Phi/(\alpha - \beta_1)$  ( $> 0$ ) is required to induce them to take the market clearing long position after a negative shock ( $\Phi < 0$ ).

## 4.2 Stabilizing rational speculation

Next, we consider an example with stabilizing speculation. The model adds another trading date to the DSSW model, at which no news occur, before the final trading date and ignores feedback effects of price changes two dates ago, which do not play a role in the DSSW model either:

**Proposition 3:** *Let  $T = 3$ ,  $\sigma_{\phi'} = 0$  (so that  $\Phi = \phi''$ ),  $t^{r''} = 1$ ,  $t^{e''} = 2$ , and  $\beta_2 = 0$ . The equilibrium asset prices are  $p_0 = v$  and*

$$p_t = v + \Phi, \quad t = 1, 2, 3, \quad (10)$$

for  $\mu > 0$  and

$$p_1 = v, \quad p_2 = v + \Phi, \quad p_3 = v + \left(1 + \frac{\beta_1}{\alpha}\right) \Phi$$

for  $\mu = 0$ .

**Proof:** From (4) with  $t^{r''} = 1$ ,  $p_1 = p_2 = p_3$  for  $\mu > 0$ . Substituting  $D_3^f = S$  and  $D_3^r = D_3^e = \alpha(v + \Phi - p_3)$  into the date-3 market clearing condition yields (10). By the same reasoning as in the proof of Proposition 1,  $p_0$  is consistent with equilibrium.

When  $\mu = 0$ ,  $D_1^f = S$  implies  $D_1^e = \alpha(v - p_1) = 0$ , so  $p_1 = v$ . At date ( $t^{e''} =$ ) 2, passive investors learn  $\Phi$  and demand  $D_2^e = \alpha(v + \Phi - p_2)$ . Since  $\Delta p_1 = 0$ , there is no feedback trading, so  $p_2 = v + \Phi$ . Solving the date-3 market clearing condition  $0 = \beta_1\Phi + \alpha(v + \Phi - p_3)$  for  $p_3$  completes the proof. ||

This example shows that rational speculation is stabilizing, rather than destabilizing, in a slightly modified version of the DSSW model (**Result 1** in the Introduction). The feedback traders go long at date 2 in response to the date-1 price increase. The difference to the DSSW model is that there is an additional trading date afterwards. As a result, no premium is required to induce rational

speculators to take the short position corresponding to the feedback traders' additional demand at date 2, because (as  $p_2 = p_3$ ) shorting is riskless. At date 3, the rational traders close the short position and leave the market, so again they carry no risk.

### 4.3 The general case

To clarify the impact of different assumptions about the timing of the arrival of news and of noise traders' feedback trades, this section considers the general model with one the signal.

**Proposition 4:** *Let  $\sigma_{\phi'} = 0$  (so that  $\Phi = \phi''$ ). For  $\mu > 0$ , the equilibrium asset prices are  $p_t = v$  for  $0, \dots, t^{r''} - 1$  and*

$$p_t = v + \left(1 + \frac{\beta_{T-t^{r''}}}{\alpha - \beta_{T-t^{r''}}}\right) \Phi, \quad t = t^{r''}, \dots, T. \quad (11)$$

*For  $\mu = 0$ , the asset prices obey  $p_t = v$  for  $0, \dots, t^{e''} - 1$  and*

$$p_t = v + \left[ \sum_{\tau=0}^{t-t^{e''}} \left(\frac{\beta}{\alpha}\right)^\tau \right] \Phi, \quad t = t^{e''}, \dots, T, \quad (12)$$

*if  $\beta_l = \beta$  is constant and*

$$p_t = v + \Phi + \frac{\beta_1}{\alpha}(p_{t-1} - p_{t-2}), \quad t = t^{e''}, \dots, T, \quad (13)$$

*if  $\beta_l = 0$  for  $l \geq 2$ .*

**Proof:** The proof is analogous to the proof of Proposition 3. ||

In the presence of rational traders, the price changes only once, viz., when the signal arrives (see (4)). From (11), whether or not the asset price overreacts and, if so, how strongly depends only on the feedback parameter  $\beta_{T-t^{r''}}$ , which determines the feedback to the single price increase at the final trading date. There is overreaction in the DSSW model, because  $\beta_{T-t^{r''}} = \beta_1 > 0$ , but not in the example of the previous section, because  $\beta_{T-t^{r''}} = \beta_2 = 0$ . Generally, if price changes further back in the past affect current feedback trading to a lesser extent than more recent price changes (i.e., if  $\beta_l$  is a decreasing sequence), then the degree of overreaction is weaker, the longer the time span  $T - t^{r''}$  between the arrival of news and the final trading date (**Result 2**).

This contrasts with the case of no rational traders: if the feedback parameters are uniform, then from (12), the *earlier* information arrives (i.e., the greater  $T - t^{e''}$ ), the stronger the price reaction. However, also from (12),

$$p_t < v + \left[ \sum_{\tau=0}^{\infty} \left(\frac{\beta}{\alpha}\right)^\tau \right] \Phi = \left(1 + \frac{\beta}{\alpha - \beta}\right) \Phi$$

for  $\Phi > 0$ . That is, even though the price does not stop rising up to the final trading date, it remains below the level to which it jumps immediately when the news arrives in the model with rational traders. As the number of trading dates grows large, the price ultimately converges to that level. Recall from Subsection 4.1 the interpretation of the date- $T$  price as reflecting fundamental risk. On this interpretation, the market with rational traders is efficient, in that the price immediately incorporates the information conveyed by  $\Phi$  and rational traders (and passive investors) get the compensation necessary to make them willing to take the investment position required for market clearing. By contrast, the asset price slowly moves into the direction of that price level if there are no rational traders. This has an important impact on how to interpret the DSSW model of Subsection 4.1: while the result that price rises above expected value is quite general, what is special and drives the result that rational speculation is destabilizing is that the model leaves no time for the asset price to rise beyond expected payoff after the arrival of the signal in the absence of rational traders (**Result 3**).

The model with no rational speculators and only one positive feedback parameter yields another interesting result: the price response to the signal is characterized by damped fluctuations (**Result 4**). With  $\beta_1$  the only positive feedback parameter, the equilibrium price sequence is determined by the second-order difference equation (13). The interesting thing is that the parameter restriction  $\beta_1 < \alpha$  is sufficient in order for the roots to be complex and stable, so that the solution to (13) is  $p_t = A(\beta_1/\alpha)^{t/2} \cos(\omega t - e)$  (where  $A$ ,  $\omega$ , and  $e$  are real numbers depending on  $\beta_1$  and  $\alpha$ ). This is consistent with the hypothesis that without rational speculators asset markets are prone to cyclical ups and downs.

## 5 Two signals

This section considers the general model with two signals. Let  $D (= \mathbf{1}_{t^{e'} < t^{r''}})$  equal one or zero, depending on whether passive investors know  $\phi'$  already when rational speculators get to know  $\phi''$  or not, respectively. Further, define  $\beta_0 = 0$ .

**Proposition 5:** *Let  $\sigma_{\phi''}^2 > 0$  and  $\mu > 0$ . The equilibrium asset prices satisfy (4) with*

$$\begin{aligned} \nu' &= \frac{\beta_{T-t^{r'}} - (\alpha - \beta_{T-t^{r''}})\lambda'}{\beta_{T-t^{r''}} - \beta_{T-t^{r'}}} \\ &= \frac{2\gamma\sigma_{\phi''}^2 \left(\frac{\alpha}{\alpha - \beta_{T-t^{r''}}}\right)^2 \left[(1 - \mu)\alpha(D - 1) + \beta_{(t^{r''}-1)-t^{r'}}\right]}{\mu + 2\gamma\sigma_{\phi''}^2 \left(\frac{\alpha}{\alpha - \beta_{T-t^{r''}}}\right)^2 \left[(1 - \mu)\alpha - \beta_{(t^{r''}-1)-t^{r'}}\right]} \end{aligned} \quad (14)$$

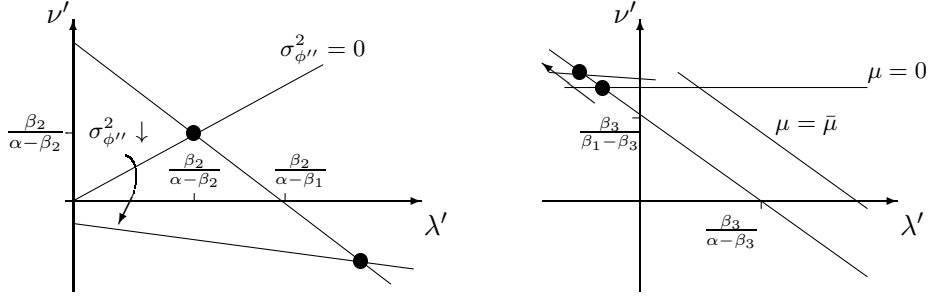


Figure 2: Determination of  $\nu'$  and  $\lambda'$ :  $T = 3$ ,  $t^{r'} = 1$ ,  $t^{e'} = 2$ ,  $t^{r''} = 2$ ,  $t^{e''} = 3$  (left panel) and  $T = 4$ ,  $t^{r'} = 1$ ,  $t^{e'} = 2$ ,  $t^{r''} = 3$  (right panel)

$$\lambda' = \frac{\mu \left[ 4\gamma\sigma_{\phi''}^2\beta_{T-t^{r''}} \left( \frac{\alpha}{\alpha-\beta_{T-t^{r''}}} \right)^2 - 1 \right]}{\mu + 2\gamma\sigma_{\phi''}^2 \left( \frac{\alpha}{\alpha-\beta_{T-t^{r''}}} \right)^2 \left[ (1-\mu)\alpha - \beta_{(t^{r''}-1)-t^{r'}} \right]} \quad (15)$$

and

$$\lambda'' = \frac{\beta_{T-t^{r''}}}{\alpha - \beta_{T-t^{r''}}} \quad (16)$$

(given that the denominators are non-zero).

**Proof:** This follows upon substituting the demands in (1), (2), and Proposition 1 into the market clearing conditions (3) for dates  $t^{r''} - 1$  and  $T$ , together with (4). Details of the algebra are in the Appendix. ||

Proposition 5 allows us to draw some general conclusions about the behavior of asset prices and is helpful in dealing with special cases.

From (11) and (16), the reaction of the asset price to the second shock is essentially the same as the response in the model with only one signal (**Result 5**). The impact of the first signal on the asset price before and after the second signal is jointly determined by (14) and (15). Suppose in what follows that the sequence of feedback parameters is decreasing, i.e.,  $\beta_{T-t^{r''}} > \beta_{T-t^{r'}}$ . Then, from (14),  $\nu'$  and  $\lambda'$  are inversely related. That is, if the price finally increases strongly after good news ( $\lambda'$  is large), then it does not rise much (or falls) at first ( $\nu'$  is small). There is overreaction to the first signal  $\phi'$  either before or after the second signal arrives (**Result 6**). This follows from (14): if  $\lambda' < 0$ , then  $\nu' > 0$ . The variance of the first signal  $\sigma_{\phi'}^2 > 0$  does not affect equilibrium prices. This is because before the arrival of the first signal both rational traders' and passive investors' asset holdings are zero, so  $\phi'$  does not affect their final wealth.

Next, consider the following special case:  $T = 3$ ,  $t^{r'} = 1$ ,  $t^{e'} = 2$ ,  $t^{r''} = 2$  (so that  $D = 0$ ), and

$t^{e''} = 3$ . Equations (14) and (15) become

$$\nu' = \frac{\beta_2 - (\alpha - \beta_1)\lambda'}{\beta_1 - \beta_2} = \frac{-(1 - \mu)2\gamma\sigma_{\phi''}^2\alpha\left(\frac{\alpha}{\alpha - \beta_1}\right)^2 - \mu\left[4\gamma\sigma_{\phi''}^2\beta_1\left(\frac{\alpha}{\alpha - \beta_1}\right)^2 - 1\right]\lambda'}{\mu + (1 - \mu)2\gamma\sigma_{\phi''}^2\alpha\left(\frac{\alpha}{\alpha - \beta_1}\right)^2}. \quad (17)$$

The left panel of Figure 2 illustrates the determination of  $\nu'$  and  $\lambda'$ . The downward-sloping line with the positive ordinate intercept depicts the function given by the first equality in (17). The other lines depict the final expression in (17) for different values of  $\sigma_{\phi''}^2$ . For  $\sigma_{\phi''}^2 = 0$ , it intersects the former line at  $(\lambda', \nu') = (\beta_2/(\alpha - \beta_2), \beta_2/(\alpha - \beta_2))$ . As  $\sigma_{\phi''}^2$  rises, the line's ordinate intercept and its slope decrease, so  $\lambda' > \beta_2/(\alpha - \beta_2) > \nu'$  for  $\sigma_{\phi''}^2 > 0$ . That is, the impact of the signal on price grows, there is no mean reversion. Surprisingly, the price may even *fall* initially in response to good news (and vice versa): for  $\sigma_{\phi''}^2$  large enough, the ordinate intercept of the curve representing the expression on the far right of (17) is less than  $-1$  and the slope is negative, so  $\nu' < -1$  and  $p_1 = v + (1 + \nu')\phi' < v$  for  $\phi' > 0$  (**Result 7**). Passive investors buy the asset at date 1 after a positive signal  $\phi' > 0$  in this case, due to the price decrease:  $D_1^e = \alpha(v - p_1) > 0$ . Since feedback traders are not yet active, this means that rational speculators go short, even though the price change  $p_2 - p_1$  contains a “large” positive predictable portion  $(\lambda' - \nu')\phi'$  (cf. (5)). To see why rational traders short the asset, rewrite (8) as

$$D_{t^{r''}-1}^r = \frac{v + \phi' - p_{t^{r''}-1}}{2\gamma(1 + \lambda'')^2\sigma_{\phi''}^2} + \left[ \frac{1}{2\gamma(1 + \lambda'')^2\sigma_{\phi''}^2} - \frac{2\alpha\lambda''}{1 + \lambda''} \right] \lambda'\phi',$$

where the second term in the square brackets represents the covariance effect encountered in Section 3. As  $\sigma_{\phi''}^2$  and  $\lambda'$  grow large, the covariance effect dominates the impact of  $\phi'$  on  $D_{t^{r''}-1}^r$ . Rational speculators anticipate the positive impact of high realizations of  $\phi''$  on the payoff of their final trading date investment and hedge their bets by shorting the risky asset at  $t^{r''} - 1$ .

The example confirms that rational speculation can be stabilizing or destabilizing. For  $\mu = 0$ , the equilibrium asset prices are

$$p_0 = p_1 = v, \quad p_2 = v + \phi', \quad p_3 = v + \left(1 + \frac{\beta_1}{\alpha}\right)\phi' + \phi''.$$

The price responds one-for-one to the current signal, but the date-3 response to the date-2 signal  $\phi'$  is  $(1 + \beta_1/\alpha)\phi'$ . So while the response to  $\phi''$  is weaker than in the presence of rational traders, the ultimate price response to  $\phi'$  is more pronounced if  $\beta_1/\alpha > \lambda'$ , which is satisfied for  $\sigma_{\phi''}^2$  small enough and  $\beta_1/\alpha > \beta_2/(\alpha - \beta_2)$ . This is in line with the result that the length of the time spell between the arrival of news and the asset payoff affects the price positively in the absence and negatively in the presence of rational traders. The example also supports the result that price

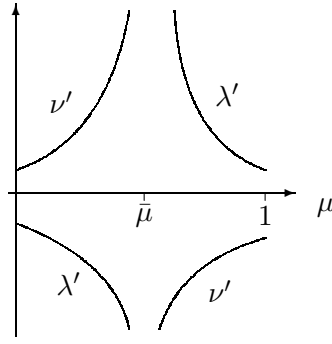


Figure 3: Impact of changes in the mass of rational speculators on the price response to the first signal

changes tend to be small when the feedback parameters are small:  $\beta_2/(\alpha - \beta_2)$  is an upper bound for  $\nu'$ , and if  $\nu' > 0$ ,  $\beta_2/(\alpha - \beta_1)$  is an upper bound for  $\lambda'$ .

As another example, let  $T = 4$ ,  $t^{r'} = 1$ ,  $t^{e'} = 2$ ,  $t^{r''} = 3$  (so that  $D = 1$ ), and  $t^{e''} = 4$ . Equations (14) and (15) become

$$\nu' = \frac{\beta_3 - (\alpha - \beta_3)\lambda'}{\beta_1 - \beta_3} = \frac{2\gamma\sigma_{\phi''}^2\beta_1\left(\frac{\alpha}{\alpha-\beta_1}\right)^2 - \mu\left[4\gamma\sigma_{\phi''}^2\beta_1\left(\frac{\alpha}{\alpha-\beta_1}\right)^2 - 1\right]\lambda'}{\mu + 2\gamma\sigma_{\phi''}^2\left(\frac{\alpha}{\alpha-\beta_1}\right)^2[(1-\mu)\alpha - \beta_1]}. \quad (18)$$

As in the previous example, the equilibrium values of  $\lambda'$  and  $\nu'$  can be illustrated by the intersection of these two straight lines (see the right panel of Figure 2). To avoid case distinctions, suppose

$$\sigma_{\phi''}^2 > \frac{1}{2\gamma\beta_1\left(\frac{\alpha}{\alpha-\beta_1}\right)^2}. \quad (19)$$

This implies that, while the denominator of the final fraction in (18) is positive for  $\mu = 0$ , it is negative for  $\mu = 1$  and that the term in square brackets in the numerator is positive. As a consequence, while the slope of the fraction is zero for  $\mu = 0$ , there exists  $\bar{\mu} \in (0, 1)$  such that the slope of the two lines in (18) is identical (viz., equal to  $-(\alpha - \beta_3)/(\beta_1 - \beta_3)$ ).

For  $\mu = 0$ , the line determined by the second equality in (18) intersects the  $\nu'$ -axis at  $\beta_1/(\alpha - \beta_1)$ , which falls short of the intercept of the first line if, and only if,  $\beta_1^2 < \alpha\beta_3$ . In that case, the intersection  $(\lambda', \nu')$  of the two lines in (18) is characterized by  $\nu' > \lambda' > 0$  for  $\mu$  sufficiently small. That is, there is mean reversion. If  $\beta_1^2 > \alpha\beta_3$ , the intersection of the lines satisfies  $\nu' > 0 > \lambda'$  for  $\mu$  sufficiently small. That is, the asset price rises above and then falls below expected value (**Result 8**).

In the DSSW model, the mass of rational investors  $\mu$  does not affect equilibrium prices (cf. DSSW, p. 388, and Proposition 2). Here, while a change in  $\mu$  does not affect the price response to the

second signal (see (16)), it does have an impact on the price response to the first signal via (14) and (15). From (19), as  $\mu$  rises and the denominator stays positive (i.e.,  $\mu < \bar{\mu}$ ), the intercept of the line determined by the second equality in (18) increases and the slope decreases, so  $\nu'$  rises and  $\lambda'$  falls. As  $\mu \rightarrow \bar{\mu}$  from below,  $\nu'$  converges to infinity and  $\lambda'$  to minus infinity. That is, for  $\mu$  less than but close enough to  $\bar{\mu}$ , the price overreacts strongly at first and then falls below expected payoff. This is the converse of the impulse response encountered in the previous example, where the asset price possibly first falls below expected payoff and then rises. This situation re-arises for  $\mu$  slightly above  $\bar{\mu}$ . As  $\mu$  rises further,  $\nu'$  rises and  $\lambda'$  falls. Thus, the impact of the first signal on the asset price is moderate both for small and for large values of  $\mu$ , but possibly dramatic for intermediate values (**Result 9**).

## 6 Conclusion

DSSW demonstrate the possibility of destabilizing rational speculation. Extending their seminal model to include additional trading dates and an additional signal yields further insights into the determination of asset prices when noise traders and rational investors interact. Overreaction to a single signal tends to be strong or weak, depending on whether the time span between the arrival of the signal and the realization of the payoff is short or long, respectively. While the same holds true for the response to the latter of two signals, the asset price may overreact, underreact, or even move in the “wrong” direction in response to the earlier signal, depending, among other things, on the proportion of rational versus passive investors. Irrespective of the shape of the impulse response functions, equilibrium asset prices can be interpreted as reflecting fundamental value. On this interpretation, the fact that price movements are less pronounced in the absence of rational speculators in the DSSW model reflects lack of time to converge to fundamental value.

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## Appendix

**Proof of Proposition 1:** Using  $D_T^r = \alpha(v + \Phi - p_T)$ , (4) for  $t = T$ , and  $p_t = p_{t+1}$  for  $t = t'', t'' + 1, \dots, T - 1$ , final wealth (6) becomes

$$W_{T+1}^r = W_{t''-1}^r + (p_{t''} - p_{t''-1})D_{t''-1}^r + \alpha(\lambda'\phi' + \lambda''\phi'' - \theta)(\lambda'\phi' + \lambda''\phi'').$$

Using (4) and independence of  $\theta$ , it follows that

$$E_{t''-1}^r W_{T+1}^r = W_{t''-1}^r + [v + (1 + \lambda')\phi' - p_{t''-1}] D_{t''-1}^r + \alpha(\lambda'^2\phi'^2 + \lambda''^2\sigma_{\phi''}^2),$$

where  $E_t^r$  denotes the expectations operator conditional on rational traders' date- $t$  information.

The conditional variance  $\sigma_{W_{T+1}|t^{r''-1}}^{r2} = E_{t^{r''-1}}^r[(W_{T+1} - E_{t^{r''-1}}^r W_{T+1})^2]$  is

$$\begin{aligned} \sigma_{W_{T+1}|t^{r''-1}}^{r2} &= E_{t^{r''-1}}^r \left[ (p_{t^{r''}} - E_{t^{r''-1}}^r p_{t^{r''}})^2 \right] D_{t^{r''-1}}^{r2} \\ &+ 2\alpha E_{t^{r''-1}}^r \left\{ (p_{t^{r''}} - E_{t^{r''-1}}^r p_{t^{r''}}) \left[ (\lambda' \phi' + \lambda'' \phi'' - \theta)(\lambda' \phi' + \lambda'' \phi'') \right. \right. \\ &\quad \left. \left. - (\lambda'^2 \phi'^2 + \lambda''^2 \phi''^2) \right] \right\} D_{t^{r''-1}}^r \\ &+ \alpha^2 E_{t^{r''-1}}^r \left\{ \left[ (\lambda' \phi' + \lambda'' \phi'' - \theta)(\lambda' \phi' + \lambda'' \phi'') - (\lambda'^2 \phi'^2 + \lambda''^2 \phi''^2) \right]^2 \right\}. \end{aligned}$$

From (4), the term in the first row on the left-hand side equals

$$(1 + \lambda'')^2 \sigma_{\phi''}^2 D_{t^{r''-1}}^{r2}.$$

Using (4), independence, and symmetry of the density of  $\phi''$  (i.e.,  $E_{t^{r''-1}}^r(\phi''^3) = 0$ ), the term in the second and third lines becomes

$$4\alpha(1 + \lambda'')\lambda'\lambda''\phi'\sigma_{\phi''}^2 D_{t^{r''-1}}^r.$$

This term captures the covariance effect mentioned in the main text. Neglecting constants, maximization of mean-variance utility  $E_{t^{r''-1}}^r W_{T+1}^r - \gamma \sigma_{W_{T+1}|t^{r''-1}}^{r2}$  is equivalent to maximizing

$$[v + (1 + \lambda')\phi' - p_{t^{r''-1}}] D_{t^{r''-1}}^r - \gamma \left[ (1 + \lambda'')^2 \sigma_{\phi''}^2 D_{t^{r''-1}}^{r2} + 4\alpha(1 + \lambda'')\lambda'\lambda''\phi'\sigma_{\phi''}^2 D_{t^{r''-1}}^r \right].$$

The expression for  $D_{t^{r''}}^r$  in the proposition is the first-order condition for this problem.

Analogously,

$$\begin{aligned} W_{T+1}^r &= W_{t^{r'-1}}^r + (p_{t^{r'}} - p_{t^{r'-1}}) D_{t^{r'-1}}^r \\ &+ (p_{t^{r''}} - p_{t^{r''-1}}) D_{t^{r''-1}}^r + \alpha(\lambda' \phi' + \lambda'' \phi'' - \theta)(\lambda' \phi' + \lambda'' \phi''). \end{aligned}$$

From (4),

$$p_{t^{r''}} - p_{t^{r''-1}} = (\lambda' - \nu')\phi' + (1 + \lambda'')\phi''.$$

From (4), and the expression for  $D_{t^{r''}}^r$ ,

$$D_{t^{r''-1}}^r = \frac{1}{1 + \lambda''} \left[ \frac{\lambda' - \nu'}{2\gamma(1 + \lambda'')\sigma_{\phi''}^2} - 2\alpha\lambda'\lambda'' \right] \phi'.$$

So

$$W_{T+1}^r = W_{t^{r'-1}}^r + (p_{t^{r'}} - p_{t^{r'-1}}) D_{t^{r'-1}}^r + \Gamma(\phi', \phi''),$$

where

$$\begin{aligned}\Gamma(\phi', \phi'', \theta) &\equiv [(\lambda' - \nu')\phi' + (1 + \lambda'')\phi''] \frac{1}{1 + \lambda''} \left[ \frac{\lambda' - \nu'}{2\gamma(1 + \lambda'')\sigma_{\phi''}^2} - 2\alpha\lambda'\lambda'' \right] \phi' \\ &\quad + \alpha(\lambda'\phi' + \lambda''\phi'' - \theta)(\lambda'\phi' + \lambda''\phi'').\end{aligned}$$

Taking expectations as of  $t^{r'} - 1$ , using  $E_{t^{r'}-1}^r p_{t^{r'}} = v$  yields

$$E_{t^{r'}-1}^r W_{T+1}^r = W_{t^{r'}-1}^r + (v - p_{t^{r'}-1})D_{t^{r'}-1}^r + E_{t^{r'}-1}^r \Gamma(\phi', \phi'', \theta).$$

The conditional variance of final wealth is

$$\begin{aligned}\sigma_{W_{T+1}^r | t^{r'}-1}^{r2} &= E_{t^{r'}-1}^r \left[ (p_{t^{r'}} - E_{t^{r'}-1}^r p_{t^{r'}})^2 \right] D_{t^{r'}-1}^{r2} \\ &\quad + 2E_{t^{r'}-1}^r \left\{ (p_{t^{r'}} - E_{t^{r'}-1}^r p_{t^{r'}}) [\Gamma(\phi', \phi'', \theta) - E_{t^{r'}-1}^r \Gamma(\phi', \phi'', \theta)] \right\} D_{t^{r'}-1}^r \\ &\quad + E_{t^{r'}-1}^r \left\{ [\Gamma(\phi', \phi'', \theta) - E_{t^{r'}-1}^r \Gamma(\phi', \phi'', \theta)]^2 \right\}.\end{aligned}$$

Using  $p_{t^{r'}} - E_{t^{r'}-1}^r p_{t^{r'}} = (1 + \nu')\phi'$  (from (4)), the term in the first row is

$$(1 + \nu')^2 \sigma_{\phi'}^2.$$

Using  $p_{t^{r'}} - E_{t^{r'}-1}^r p_{t^{r'}} = (1 + \nu')\phi'$  and the fact that  $\phi'$  has mean zero, the term in the second row can be written as

$$2(1 + \nu')E_{t^{r'}-1}^r \left\{ \phi' \Gamma(\phi', \phi'', \theta) \right\} D_{t^{r'}-1}^r$$

or, inserting the definition of  $\Gamma(\phi', \phi'', \theta)$ ,

$$\begin{aligned}2(1 + \nu') &\left\{ \left[ \frac{\lambda' - \nu'}{2\gamma(1 + \lambda'')\sigma_{\phi''}^2} - 2\alpha\lambda'\lambda'' \right] \left[ \frac{\lambda' - \nu'}{1 + \lambda''} E_{t^{r'}-1}^r (\phi'^3) + E_{t^{r'}-1}^r (\phi'^2 \phi'') \right] \right. \\ &\quad + \alpha \left[ \lambda'^2 E_{t^{r'}-1}^r (\phi'^3) + \lambda''^2 E_{t^{r'}-1}^r (\phi' \phi''^2) + 2\lambda'\lambda'' E_{t^{r'}-1}^r (\phi'^2 \phi'') \right. \\ &\quad \left. \left. - \lambda' E_{t^{r'}-1}^r (\theta \phi') - \lambda'' E_{t^{r'}-1}^r (\theta \phi'') \right] \right\} D_{t^{r'}-1}^r.\end{aligned}$$

Because of symmetry and independence, all expected values equal zero. So, neglecting constants, maximization of  $E_{t^{r'}-1}^r W_{T+1}^r - \gamma \sigma_{W_{T+1}^r | t^{r'}-1}^{r2}$  is equivalent to maximizing

$$(v - p_{t^{r'}-1})D_{t^{r'}-1}^r - \gamma(1 + \nu')^2 \sigma_{\phi'}^2 D_{t^{r'}-1}^{r2}.$$

The expression for  $D_{t^{r'}}^r$  in the proposition is the first-order condition for this problem. ||

**Proof of Proposition 5:** From (4), the price changes twice, viz., at  $t^{r'}$  and  $t^{r''}$ . So at date  $T$ , there is positive feedback to the price increases  $T - t^{r'}$  and  $T - t^{r''}$  dates ago, and the market clearing condition becomes

$$0 = \beta_{T-t^{r''}}(p_{t^{r''}} - p_{t^{r''}-1}) + \beta_{T-t^{r'}}(p_{t^{r'}} - p_{t^{r'}-1}) + \alpha(v + \phi' + \phi'' - p_T).$$

Solving for  $p_T$  and eliminating the past prices using (4) yields

$$p_T = v + \left[ 1 + \frac{\beta_{T-t^{r'}} + (\beta_{T-t^{r'}} - \beta_{T-t^{r''}})\nu'}{\alpha - \beta_{T-t^{r''}}} \right] \phi' + \left( 1 + \frac{\beta_{T-t^{r''}}}{\alpha - \beta_{T-t^{r''}}} \right) \phi''.$$

This proves (14) and (16). From (1), (2), (4), and Proposition 1, market clearing at date  $t^{r''} - 1$  implies

$$\begin{aligned} 0 = & \beta_{(t^{r''-1})-t^{r'}}(p_{t^{r''-1}} - v) + \mu \left[ \frac{v + (1 + \lambda')\phi' - p_{t^{r''-1}}}{2\gamma(1 + \lambda'')^2\sigma_{\phi''}^2} - \frac{2\alpha\lambda'\lambda''\phi'}{1 + \lambda''} \right] \\ & + (1 - \mu)\alpha(v + D\phi' - p_{t^{r''-1}}) \end{aligned}$$

Solving for  $p_{t^{r''-1}}$ , substituting for  $\lambda''$  from (16), and rearranging terms gives

$$\begin{aligned} p_{t^{r''-1}} = & v + \left\{ 1 + \frac{2\gamma\sigma_{\phi''}^2 \left( \frac{\alpha}{\alpha - \beta_{T-t^{r''}}} \right)^2 \left[ (1 - \mu)\alpha(D - 1) + \beta_{(t^{r''-1})-t^{r'}} \right]}{\mu + 2\gamma\sigma_{\phi''}^2 \left( \frac{\alpha}{\alpha - \beta_{T-t^{r''}}} \right)^2 \left[ (1 - \mu)\alpha - \beta_{(t^{r''-1})-t^{r'}} \right]} \right. \\ & \left. + \frac{\mu \left[ 1 - 4\gamma\sigma_{\phi''}^2\beta_{T-t^{r''}} \left( \frac{\alpha}{\alpha - \beta_{T-t^{r''}}} \right)^2 \right] \lambda'}{\mu + 2\gamma\sigma_{\phi''}^2 \left( \frac{\alpha}{\alpha - \beta_{T-t^{r''}}} \right)^2 \left[ (1 - \mu)\alpha - \beta_{(t^{r''-1})-t^{r'}} \right]} \right\} \phi'. \end{aligned}$$

This proves (15). The fact that  $p_0 = v$  is an equilibrium price follows from the same reasoning as in the proof of Proposition 2. ||