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Listening to the Noise in Financial Markets

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Abstract

Do all types of information benefit the efficiency of prices in the sense that they drive them closer to fundamentals compared to the situation where information does not exist? Looking at the competitive noisy rational expectations framework, the clear answer of the literature is: yes. It suggests that rational traders use all available types of information to submit more sophisticated market orders, thereby boosting price efficiency. In this paper, however, we propose a contradiction to this traditional view. We show that there exist types of non-fundamental information that are detrimental to price efficiency, as they lead traders to rationally trade with rather than against noise. We develop an analytically tractable framework with public non-fundamental information and prove that this type of information can harm price efficiency, i.e., prices would be closer to fundamentals if public non-fundamental information did not exist.

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Keywords: Rational Expectations Equilibrium, Market Efficiency, Non-Fundamental Information, Destabilizing Rational Speculation.

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1 Introduction

This paper shows that information about future noise trader demand potentially *reduces* the informativeness of current asset prices in dynamic noisy rational expectations equilibrium (REE). That is, contrary to public information about noise in a static setting and contrary to public information about fundamentals in a dynamic setting, public information about investor sentiment may be harmful to the accuracy of asset prices as predictors of fundamental value.

Asset prices aggregate investors' dispersed information in REE. The asset price reveals agents' private information imperfectly if the asset supply is stochastic (see Grossman and Stiglitz, 1980; Hellwig, 1980; Verrecchia, 1982). A common interpretation of stochastic supply faced by rational traders is stochastic noise trader demand (see, e.g., Black, 1986). One of the central questions of empirical behavioural finance is how noise trader demand (i.e., investor sentiment) in financial markets can be measured. The literature mainly distinguishes between three types.[1] The two traditional types try to gauge investor sentiment by relying on market data such as total trading volume and closed-end fund discounts (see, e.g., Lee et al. (1991) and Baker and Wurgler (2006)) and on survey data (see, e.g., Clarke and Statman (1998) and Brown and Cliff (2005)), respectively.

More recently, a third type that takes data from publicly available media content such as internet search results, blogs, and stock message boards has gained enormous popularity among empirical researchers. Pioneering work in this area has been done by Wysocki (1998), who shows that the cumulative stock message posting volume on Yahoo! predicts shifts in next day stock returns. Antweiler and Frank (2004) and Das and Chen (2007) prove that the sentiment derived from stock message boards is linked to volatility and trading volume. Karabulut (2013) shows that Facebook's Gross National Happiness index predicts stock market returns and trading volume.

Da et al. (2015) measure investor sentiment through constructing a Financial and Economics Attitudes Revealed by Search (FEARS) index, that is based on internet search behaviour of private households. A higher FEARS index predicts increasing stock market returns in the next two days. Moreover, it is correlated to changes in volatility and mutual fund flows out of equity funds. Sul et al. (2017) show that the sentiment contained in the tweets from StockTwits predicts stock returns. Along similar lines, Duz and Tas (2020) conclude that

[1] An excellent overview is given by Zhou (2018).

the content of StockTwits has a strong predictive power for short-term price movements and should play a role in rational traders' (short) investment strategies.

Though attracting a lot of attention from empirical researchers, there is no contribution in the literature so far that explores the effects of such information about investor sentiment from a theoretical perspective. Since media contents such as internet research results and stock message boards are publicly accessible, such information can be interpreted as a publicly available proxy for the mood and the behaviour of noise traders, i.e., as noisy public non-fundamental information. Due to their strong predictive power, media contents do not only reveal public information about the *contemporaneous* behaviour of noise traders, but also about their *future* behaviour in the market. That is, today's sentiment on stock message boards serves as a proxy for how noise traders act tomorrow and shape tomorrow's prices in financial markets.

The aim of this paper is to explore the effects of public non-fundamental information on price informativeness, i.e., on how efficiently prices reflect their underlying fundamentals. We build on the competitive noisy rational expectations framework in the spirit of Grossman and Stiglitz (1980). Within this framework, the common result is that both available fundamental and non-fundamental information boosts the efficiency of prices, i.e., prices are always more efficient if there exists information compared to the situation without information.

Building on Allen et al. (2006), Gao (2008) explores the effects of (noisy) public *fundamental* information on price informativeness when traders exhibit a short trading horizon. In this situation, traders are concerned with forecasting next period's price rather than fundamentals and, thus, with forecasting others' expectations (that determine next period's price). Hence, short-term traders tend to overweight public (noisy) information in their demand as they know that next period's traders observe exactly the same public information as they do. Thus, it significantly helps them to forecast next period's traders' expectations. The main concern in this context is that the existence of noisy public fundamental information can harm the efficiency of prices in the sense that it biases them away from fundamentals towards the value of the noisy public information. Gao (2008), however, dispels this concern by unambiguously proving that public fundamental information increases price informativeness, even if traders exhibit a short trading horizon.

Turning to the case of non-fundamental information, the results of the so far existing literature unequivocally state that the existence of non-fundamental information boosts the efficiency of prices in a competitive trading environment. Though focusing on equilibrium multiplicity and complementarities in information acquisition, Ganguli and Yang (2009) and Manzano and Vives (2011) expound in a static economy that the existence of private non-fundamental information always leads to more informative prices in financial markets compared to the situation where it does not exist. Marmora and Rytchkov (2018) assign traders a fixed information processing capacity to produce private fundamental and private non-fundamental information. They unambiguously show that the existence of non-fundamental information increases price informativeness although it can crowd out fundamental information. Along similar lines, Farboodi and Veldkamp (2020) analyze the effects coming from non-fundamental information acquisition in an infinite-horizon economy that is populated by overlapping generations and characterized by exogenous technical progress over time. They assign traders a data constraint to process contemporaneous private fundamental and private non-fundamental information. Analogously to Marmora and Rytchkov (2018), they prove that non-fundamental information also increases price informativeness in a dynamic context.

In strong contrast to all described results, we show that public non-fundamental information can harm the efficiency of prices, i.e., there exist situations in which prices would be more informative if non-fundamental information was absent. We present analytically tractable variants of a three-period, competitive economy with contemporaneous and forward-looking public non-fundamental information where traders exhibit a long-term and a short-term horizon, respectively. In the long-lived agents (LLA) model, future public information unambiguously harms price informativeness if public information about contemporaneous noise trader demand is absent. It can also lower price efficiency if contemporaneous public information is present. In the short-lived agents (SLA) model, public information about future noise demand can lower price informativeness, too.

The driving force behind these results lies in the fact that traders in the first period rationally trade *with* rather than against their signal about second period's noise trader demand. If they a receive a higher signal, they raise their demand in expectation of a higher price in the second second. If their signal becomes more precise, they tend to put more weight on it in their demand for the asset, which tends to make prices less efficient in the first period. However, since information is public, there is an additional, stabilizing effect. Traders in the second period exclusively use public information about second-period noise trader demand to extract noise from the second-period price to gain more precise information about fundamentals out of it. As the signal becomes more precise, they counteract more of the influence of noise traders, which reduces their effect on the second-period price. This, in return, makes the public signal less valuable from the perspective of first-period traders, which leads them to put less weight on it when forming their demand. As we show, the former effect especially dominates when public fundamental information is imprecise, thereby leading to the conclusion that prices would be more efficient if information was absent.

In the SLA model, public information about future noise trader demand plays an additional, stabilizing role by weakening the well-known Keynesian beauty contest. The more precise the signal, the more noise is counteracted by rational traders in the second period and, thus, the closer the price is driven towards fundamentals. This fact allows traders in the first period to trade more aggressively on their private fundamental information, thereby tending to boost price efficiency in the first period. Nevertheless, future public non-fundamental information can harm price informativeness in the SLA model if it is sufficiently imprecise.

The outcomes of our paper relate to the small literature on destabilizing rational speculation in financial markets. If noise traders are modelled as positive feedback traders, rational investors can systematically drive prices away from fundamentals by anticipating the behaviour of positive feedback traders. If rational traders increase their demand for the asset, positive feedback traders will raise theirs as a reaction to higher prices. This pushes up prices even further, which allows rational traders to sell the asset at a higher price in the future (see De Long et al. (1990) and Arnold and Brunner (2015)). In a setup à la Kyle (1985), which implies a frictionless market and strategic behaviour, Madrigal (1996) shows that the existence of a non-fundamental speculator can harm price efficiency. The existence of the speculator makes the fundamentally informed trader convey less information to limit the speculator's profit.[2] In a recent contribution and also using the Kyle (1985) setup, Sadzik and Woolnough (2020) show that a trader with information about a persistent component of noise trader demand can destabilize prices by amplifying its impact on them. However, the results obtained in the Kyle (1985) setup can hardly be compared to our ones as the two frameworks significantly differ from each other. Our contribution is to show that there are types of information in a competitive market whose existence can harm the efficiency of prices. That is, prices would be closer to fundamentals if information was unavailable.

The remainder of this paper is structured as follows: Section 2 develops a static benchmark.

[2]The original paper contains some errors, that have been corrected by Yang and Zhu (2017).

Section 3 describes the LLA model. In Section 4, we turn to the SLA model. Section 5 concludes.

2 Static model

This section shows that in the standard static REE setup noisy public information about non-rational traders' asset demand increases the informativeness of prices.

2.1 Model

Consider a static one-good economy populated by a unit mass of rational investors indexed by the interval $[0, 1]$ and by a set of noise traders. Rational agents value consumption π according to the CARA utility function $-\exp(-\gamma^{-1}\pi)$, where γ (> 0) is their risk tolerance (i.e., the inverse of the degree of absolute risk aversion). There are one risky and one safe asset. The supply of the risky asset is fixed and normalized to zero for simplicity. A risky asset pays off θ units of consumption. The safe asset is in perfectly elastic supply. Its safe rate of return is zero. Initial endowments are also normalized to zero. Agents trade the assets in the financial market. Noise trader demand for the risky asset s is exogenous. Rational agents maximize expected utility conditional on available information. Agent i obtains a private signal $x_i = \theta + \varepsilon_i$ about θ. In addition, rational traders receive a public signal for noise trader demand

$$Y = s + \eta. \tag{1}$$

The random variables $\xi \in \{\theta, s, \varepsilon_i, \eta \,|\, i \in [0, 1]\}$ are jointly normally and independently distributed. The means of these variables are all normalized to zero. The precision of ξ is denoted τ_ξ (i.e., the variance of ξ is τ_ξ^{-1}).

2.2 Rational expectations equilibrium

Suppose the price of the risky asset is a linear function of θ, s, and η:

$$P = a\theta + bs - cY \tag{2}$$

for constants a, b, and c. Investor i extracts information about the asset payoff θ from the two signals she receives as well as from the asset price. The vector of her signals is denoted

$I_i = (P, x_i, Y)$. Her investment in the risky asset and her final wealth are denoted D_i and $\pi_i = (\theta - P)D_i$, respectively.

Definition (rational expectations equilibrium): A price function (2) and asset demands D_i, $i \in [0,1]$, are a linear *rational expectations equilibrium (REE)* if D_i maximizes expected utility $\mathrm{E}[-\exp(\gamma^{-1}\pi_i) \,|\, I_i]$ for all $i \in [0,1]$ and the market for the risky asset clears, i.e., $\int_0^1 D_i di + s = 0$.

Expected utility maximization yields the asset demands

$$D_i = \gamma \, \frac{\mathrm{E}(\theta \,|\, I_i) - P}{\mathrm{var}(\theta \,|\, I_i)}. \tag{3}$$

Positive signals received by other agents raise their demands. So the asset price P contains valuable information about fundamentals θ. The public signal about noise Y does not correlate with θ, but it helps to disentangle the impacts of the private signals about fundamentals on the one hand and noise trader demand on the other hand on P. Define

$$P^* \equiv \frac{P + cY}{a} - \frac{b}{a}\mathrm{E}(s \,|\, Y). \tag{4}$$

From $\mathrm{E}(s \,|\, Y) = \tau_\eta Y/(\tau_s + \tau_\eta)$ and (1),

$$P^* = \theta + \frac{1}{\rho}\frac{\tau_s s - \tau_\eta \eta}{\tau_s + \tau_\eta},$$

where $\rho \equiv a/b$. That is, P^* is a signal about θ with precision $\rho^2(\tau_s + \tau_\eta)$. It aggregates the information contained in P and Y: $\mathrm{E}(\theta \,|\, I_i) = \mathrm{E}(\theta \,|\, P^*, x_i)$.[3] An increase in the precision of the signal about noise trader demand allows a more accurate prediction of θ. The conditional moments in i's asset demand function (3) are

$$\begin{aligned} \mathrm{E}(\theta \,|\, I_i) &= \frac{\tau_\varepsilon x_i + \rho^2(\tau_s + \tau_\eta)P^*}{\tau_\theta + \tau_\varepsilon + \rho^2(\tau_s + \tau_\eta)} \\ \mathrm{var}(\theta \,|\, I_i) &= \frac{1}{\tau_\theta + \tau_\varepsilon + \rho^2(\tau_s + \tau_\eta)}. \end{aligned}$$

Inserting the demands into the asset market clearing condition, applying the strong law of large numbers (i.e., $\int_0^1 \varepsilon_i = 0$), using (4) and $\mathrm{E}(s \,|\, Y) = \tau_\eta Y/(\tau_s + \tau_\eta)$, and solving for P yields

$$P = \frac{a\tau_\varepsilon \theta + a\gamma^{-1}s + [\rho^2(\tau_s + \tau_\eta)c - a\rho\tau_\eta]\, Y}{a(\tau_\theta + \tau_\varepsilon) + (a-1)\rho^2(\tau_s + \tau_\eta)}.$$

[3] This also follows from the projection theorem.

Matching coefficients with (2) yields:

Proposition 1: *There exists a unique linear REE, with*

$$\rho = \gamma\tau_\varepsilon \tag{5}$$

$$a = \frac{\tau_\varepsilon + \rho^2(\tau_s + \tau_\eta)}{\tau_\theta + \tau_\varepsilon + \gamma^2\tau_\varepsilon^2(\tau_s + \tau_\eta)} \tag{6}$$

$$b = \frac{a}{\gamma\tau_\varepsilon} \tag{7}$$

$$c = \frac{a\rho\tau_\eta}{\tau_\varepsilon + \rho^2(\tau_s + \tau_\eta)}. \tag{8}$$

2.3 Price informativeness

The central question we are interested in is: how does the precision of the public signal about noise τ_η affect the informativeness of the asset price as measured by $\mathrm{var}^{-1}(\theta \mid P)$?[4] The unambiguous answer of the static model is: the informativeness of the asset price is higher with a more precise signal. That is, if one adds a single rational investor who does not receive any signal to the model, the asset price allows that investor to make a more informed investment decision and obtain a higher level of expected utility.[5]

Observing P is informationally equivalent to observing $P^{**} = P/a$. From (2),

$$P^{**} = \theta + \frac{1}{\rho}s - \frac{c}{a}Y.$$

Using (1), the variance of P^{**} conditional on θ can be written as

$$\mathrm{var}(P^{**} \mid \theta) = \left(\frac{1}{\rho} - \frac{c}{a}\right)^2 \frac{1}{\tau_s} + \left(\frac{c}{a}\right)^2 \frac{1}{\tau_\eta}.$$

The signal Y about noise trader demand affects the conditional price variance in two ways. On the one hand, it dampens the impact of the noise shock s on the price (via the term $-(c/a)\tau_s^{-1}$). On the other hand, the disturbance of the signal η acts as an additional source of noise in equilibrium (captured by the term $-(c/a)\tau_\eta^{-1}$). Following Gao (2008), we call these dual roles of information about noise trader demand the *information role* and the

[4] Vives (2008, 121) calls $\mathrm{var}^{-1}(\theta \mid P)$ "price precision".

[5] The same holds true for a set of measure zero of investors and, by continuity, for a positive but sufficiently small mass of investors.

commonality role, respectively.[6] From $\text{var}^{-1}(\theta|\, P^{**}) = \tau_\theta + \text{var}^{-1}(P^{**}|\, \theta)$, using the coefficients in Proposition 1, we get

$$\text{var}^{-1}(\theta|\, P^{**}) = \tau_\theta + \frac{\rho^2 \tau_s \left[1 + \rho\gamma(\tau_s + \tau_\eta)\right]^2}{1 + \rho\gamma\tau_s \left[2 + \rho\gamma(\tau_s + \tau_\eta)\right]}.$$

It is easily checked that the fraction on the right-hand side is an increasing function of $2 + \rho\gamma(\tau_s + \tau_\eta)$. This proves:

Proposition 2: *An increase in the precision of the signal about noise trader demand raises price informativeness:*

$$\frac{\partial \left[\text{var}^{-1}(\theta|\, P^{**})\right]}{\partial \tau_\eta} > 0.$$

That is, in a static setting the information role of public non-fundamental information dominates the commonality role: noisy information about non-rational traders' demand makes prices unequivocally more informative about fundamental value.

3 Dynamic model

This section introduces an additional trading period to the model of Section 2. In the resulting dynamic model information traders receive about *future* noise trader demand can have a negative impact on the current informativeness of prices.

3.1 Model

There are now two trading dates before the assets pay off. A unit mass of rational investors indexed by the interval $[0, 1]$ and a set of noise traders trade at both dates. Rational agents value consumption according to the same CARA utility function as before. There are a fixed supply equal to zero of a risky asset that pays off θ and a perfectly elastic supply of a safe asset with a zero rate of return. Noise trader demand s_t at trading date t $(= 1, 2)$ is exogenous. Rational agent i receives three signals at date 1: a private signal $x_i = \theta + \varepsilon_i$ about fundamentals and public signals

$$Y_t = s_t + \eta_t, \quad t = 1, 2. \tag{9}$$

[6] In Gao (2008), these terms refer to a public signal about fundamentals, not about noise trader demand.

We can thus distinguish between signals about contemporaneous and about future noise trader demand. The random variables $\xi \in \{\theta, s_t, \varepsilon_i, \eta_t \,|\, t = 1, 2,\ i \in [0,1]\}$ are jointly normally and independently distributed. The means are all normalized to zero. The precision of ξ is denoted τ_ξ.

3.2 Equilibrium

Let P_t denote the asset price at trading date t $(= 1, 2)$. Suppose the asset prices obey

$$\begin{aligned} P_1 &= a_1\theta + b_1 s_1 - c_{11} Y_1 + c_{12} Y_2 & (10) \\ P_2 &= a_2\theta + b_2 s_2 - c_{21} Y_1 - c_{22} Y_2 + d_2 P_1. & (11) \end{aligned}$$

for constants a_t, b_t, c_{1t}, c_{2t}, and d_2. Let $I_{i1} = (P_1, x_i, Y_1, Y_2)$ and $I_{i2} = (P_1, x_i, Y_1, Y_2, P_2)$ denote the vectors of signals available to i at dates 1 and 2, respectively. D_{i1} and D_{i2} denote her asset demands at the two trading dates, and $\pi_i = (P_2 - P_1)D_{1i} - (\theta - P_2)D_{i2}$ is her final wealth.

Definition (dynamic rational expectations equilibrium): Price functions (10) and (11) and asset demands D_{it}, $t = 1, 2$, $i \in [0, 1]$, are a linear *dynamic rational expectations equilibrium (REE)* if D_{i2} maximizes date-2 expected utility $\mathrm{E}[-\exp(\gamma^{-1}\pi_i) \,|\, I_{i2}]$ and D_{i1} maximizes date-1 expected utility $\mathrm{E}[-\exp(\gamma^{-1}\pi_i) \,|\, I_{i1}]$ given D_{i2} for all $i \in [0, 1]$ and the market for the risky asset clears at both trading dates, i.e., $\int_0^1 D_{it} di + s_t = 0$, $t = 1, 2$.

The utility maximizing demands are given by Proposition A3 in Brown and Jennings (1989, 544) and Proposition B.1 in Avdis (2016, 579):

$$\begin{aligned} D_{i2} &= \gamma \frac{\mathrm{E}(\theta \mid I_{i2}) - P_2}{\mathrm{var}(\theta \mid I_{i2})} & (12) \\ D_{i1} &= \gamma \frac{\mathrm{E}\left[P_2 - h(\theta - P_2) \mid I_{i1}\right] - P_1}{\mathrm{var}\left[P_2 - h(\theta - P_2) \mid I_{i1}\right]}, & (13) \end{aligned}$$

where

$$h = \frac{\mathrm{cov}(\theta - P_2, P_2 \mid I_{i1})}{\mathrm{var}(\theta - P_2 \,|I_{i1})}. \tag{14}$$

They result from solving the problem backwards. Demand at the final trading date (12) is analogous to (3). It depends on the conditional moments of the asset's payoff θ and on the purchase price P_2. Substituting D_{i2} into $\pi_i = (P_2 - P_1)D_{1i} - (\theta - P_2)D_{i2}$ and the resulting expression for π_i into $\mathrm{E}[-\exp(\gamma^{-1}\pi_i) \,|\, I_{i1}]$ yields i's expected utility as of date 1 as a function of D_{i1}. The return on date-1 investments is given, not by asset fundamentals, but by the resale price P_2. However, P_2 also affects the net payoff $\theta - P_2$ on date-2 investments D_{i2}. As a result, D_{i1} depends on the moments of $P_2 - h(\theta - P_2)$. A decrease in the covariance of P_2 and $\theta - P_2$ raises the expected profitability of date-1 investments in the risky asset (see (13) and (14)).

As in the static model, even though the signals Y_t about the noise trader demands do not correlate with θ, they can be used to disentangle the impacts of the private fundamental signals and noise trader demand on the asset price. Let

$$\begin{aligned} P_1^* &\equiv \frac{P_1 + c_{11}Y_1 - c_{12}Y_2}{a_1} - \frac{b_1}{a_1}\mathrm{E}(s_1 \,|\, Y_1) \\ P_2^* &\equiv \frac{P_2 + c_{21}Y_1 + c_{22}Y_2 - d_2P_1}{a_2} - \frac{b_2}{a_2}\mathrm{E}(s_2 \,|\, Y_2). \end{aligned}$$

Using $\rho_t \equiv a_t/b_t$, $\mathrm{E}(s_t \,|\, Y_t) = \tau_{\eta t}Y_t/(\tau_{st} + \tau_{\eta t})$, and (9), it follows that P_t^* is a signal about θ with precision $\rho_t^2(\tau_{st} + \tau_{\eta t})$:

$$P_t^* = \theta + \frac{1}{\rho_t}\frac{\tau_{st}s_t - \tau_{\eta t}\eta_t}{\tau_{st} + \tau_{\eta t}}.$$

(P_1^*, x_i) and (P_1^*, x_i, P_2^*) convey the same information as I_{i1} and I_{2i}, respectively.[7]

The equilibrium analysis starts with date 2. Exploiting the information content of the signals in (P_1^*, x_i, P_2^*), one obtains the conditional moments in (12). Substitution of the D_{2i}'s into the date-2 market clearing condition yields P_2 as linear function of θ, s_2, Y_1, Y_2, and P_1. The coefficients of these variables are matched with those in (11). Going back to date 1, one computes the conditional moments in (13), given the linear price functions (10) and (11). The date-1 market clearing condition yields P_1 as a linear function of θ, s_1, Y_1, and Y_2, whose coefficients are matched with those in (10). The resulting system of equations allows for a closed-form solution of the coefficients in (10) and (11).

[7] Again this follows from the projection theorem.

Proposition 3: *There exists a unique linear dynamic REE, with*[8]

$$
\begin{aligned}
\rho_1 &= \gamma\tau_\varepsilon \\
\rho_2 &= \gamma\tau_\varepsilon \\
\Delta &= \tau_\theta + \tau_\varepsilon + \rho_1^2(\tau_{s1} + \tau_{\eta 1}) + \rho_2^2(\tau_{s2} + \tau_{\eta 2}) \\
a_2 &= \frac{\tau_\varepsilon + \rho_2^2(\tau_{s2} + \tau_{\eta 2})}{\Delta} \\
b_2 &= \frac{1 + \gamma\rho_2(\tau_{s2} + \tau_{\eta 2})}{\gamma\Delta} \\
\Gamma_1 &= \left[\tau_\theta + \tau_\varepsilon + \rho_1^2(\tau_{s1} + \tau_{\eta 1})\right]^{-1} \\
\Gamma_2 &= (\tau_{s2} + \tau_{\eta 2})^{-1} \\
h &= \frac{a_2(1-a_2)\Gamma_1 - b_2^2\Gamma_2}{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2} \\
a_1 &= \frac{\left[\tau_\varepsilon + \rho_1^2(\tau_{s1} + \tau_{\eta 1})\right]\Gamma_1\Gamma_2 b_2^2}{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2} + (1+h)\frac{\rho_1^2(\tau_{s1} + \tau_{\eta 1})}{\Delta} \\
b_1 &= \frac{a_1}{\rho_1} \\
c_{11} &= a_1 \frac{\rho_1\tau_{\eta 1}\left(1 + \frac{1-a_2}{\Delta b_2^2\Gamma_2}\right)}{\tau_\varepsilon + \rho_1^2(\tau_{s1} + \tau_{\eta 1})\left(1 + \frac{1-a_2}{\Delta b_2^2\Gamma_2}\right)} \\
c_{12} &= a_1 \frac{\frac{1-a_2}{b_2}\tau_{\eta 2}\left(1 - \frac{\rho_2}{b_2\Gamma_2\Delta}\right)}{\tau_\varepsilon + \rho_1^2(\tau_{s1} + \tau_{\eta 1})\left(1 + \frac{1-a_2}{b_2^2\Gamma_2\Delta}\right)} \\
c_{21} &= \frac{-\rho_1^2(\tau_{s1} + \tau_{\eta 1})\frac{c_{11}}{a_1} + \rho_1\tau_{\eta_1}}{\Delta} \\
c_{22} &= \frac{\rho_1^2(\tau_{s1} + \tau_{\eta 1})\frac{c_{12}}{a_1} + \rho_2\tau_{\eta 2}}{\Delta} \\
d_2 &= \frac{\rho_1^2(\tau_{s1} + \tau_{\eta 1})}{a_1\Delta}.
\end{aligned}
$$

3.3 Price informativeness

The static model of Section 2 sheds light on the impact of public information about *contemporaneous* noise trader demand on the informativeness of asset prices. The dynamic model

[8]The equations provide a closed-form solution because whenever a variable appears that is not a primitive of the model, it is determined in one of the preceding equations. Explicit substitution would create very messy expressions.

of the present section allows an assessment of information about *future* noise trader demand. This subsection shows that type of public information can reduce price informativeness. Define $P_1^{**} = P_1/a_1$:

$$P_1^{**} \equiv \theta + \frac{1}{\rho_1}s_1 - \frac{c_{11}}{a_1}Y_1 + \frac{c_{12}}{a_1}Y_2.$$

Substituting for Y_t from (9), we get

$$\operatorname{var}^{-1}(\theta \mid P_1^{**}) = \tau_\theta + \left[\left(\frac{1}{\rho_1} - \frac{c_{11}}{a_1}\right)^2 \frac{1}{\tau_{s1}} + \left(\frac{c_{11}}{a_1}\right)^2 \frac{1}{\tau_{\eta 1}} + \left(\frac{c_{12}}{a_1}\right)^2 \left(\frac{1}{\tau_{s2}} + \frac{1}{\tau_{\eta 2}}\right)\right]^{-1}, \quad (15)$$

where the term in square brackets is $\operatorname{var}(P_1^{**} \mid \theta)$. The crucial question is: what is the impact of the precision of the public signal about date-2 noise trader demand $\tau_{\eta 2}$ on the informativeness of the equilibrium date-1 asset price.

A remarkable feature of the model is that price informativeness is generally higher with a completely uninformative signal about future noise trader demand than with a perfectly informative signal:

Proposition 4: $\operatorname{var}^{-1}(\theta \mid P_1^{**})$ *is greater for* $\tau_{\eta 2} \to 0$ *than for* $\tau_{\eta 2} \to \infty$

The proof is in the Appendix. Forward-looking public non-fundamental information does not influence the first-period price in both limiting cases (i.e., $c_{12} = 0$ if $\tau_{\eta 2} \to 0$ and $\tau_{\eta 2} \to \infty$). If Y_2 is completely imprecise, agents, of course, refrain from trading on it. If it is perfectly precise, traders in the second period offset all noise inherent in the price (i.e., $b_2 = c_{22}$). P_2 equals θ and noise trading does not shape the price in the second period anymore. This makes Y_2 completely useless for predicting P_2. Additionally, ρ_1 is independent of $\tau_{\eta 2}$ (see proposition 3). This means that the precision of public non-fundamental does not affect how aggressively agents trade on their private fundamental information in the first period.

As a consequence, the result in proposition 4 is driven by the fraction c_{11}/a_1, i.e., by how aggressively agents trade on contemporaneous public non-fundamental information compared to private fundamental information (where the latter is independent of $\tau_{\eta 2}$). Agents trade less aggressively on Y_1 if forward-looking public non-fundamental information is perfectly precise than if it is completely imprecise. Apart from extracting noise from P_1 to gain a more precise signal for fundamentals, traders in the first period use Y_1 to predict P_2 as the public information can also be observed in the second period. However, if Y_2 is perfectly precise, θ can be perfectly observed in the second period by disentangling the price. As a consequence, traders in the second period do not use Y_1 to predict fundamentals as P_2 together with Y_2

already reveals its value. This makes Y_1 completely useless for predicting P_2 and traders put less weight on it in the first period. By contrast, if forward-looking information is completely imprecise, traders in the second period use Y_1 together with P_1 to update their beliefs about fundamentals. This leads traders in the first period to additionally use Y_1 for predicting next period's price. This makes them put more weight on the signal.

However, the effect from more aggressive trading on Y_1 on price informativeness is two-edged. On the one hand, more noise coming from s_1 is offset, which represents the stabilizing information role of contemporaneous public information. On the other hand, the effect of the common error term in Y_1 is amplified, thereby representing the destabilizing commonality role of public information. Consequently, both opposed effects are more pronounced at $\tau_{\eta 2} \to 0$ than at $\tau_{\eta 2} \to \infty$. Nevertheless, the result in proposition 4 unambiguously shows that the increase in the stabilizing information role dominates the increase in the destabilizing commonality role. Therefore, the price in the first period is more informative if Y_2 is absent.

Next, suppose there is valuable information only about future, and not about contemporaneous, noise trader demand. In this case information about date-2 noise trader demand is unequivocally harmful to price informativeness at date 1:

Proposition 5: *Let* $\tau_{\eta 1} = 0$. *Then* $\mathrm{var}^{-1}(\theta \mid P_1^{**})$ *is maximum for* $\tau_{\eta 2} = 0$.

The proof is in the Appendix. The fact that Y_1 has zero precision means that it does not affect P_1, so the c_{11}-terms drop out of (16) (cf. (10)). For $\tau_{\eta 2} = 0$, the expression for c_{12}/a_1 in Proposition 3 is also zero. The assertion follows from $(c_{12}/a_1)^2 > 0$ for all $\tau_{\eta 2} > 0$ and the fact that the equilibrium value of ρ_1 in Proposition 3 is independent of $\tau_{\eta 2}$. Actually, c_{12}/a_1 converges to zero as $\tau_{\eta 2} \to \infty$, so the relation between $\mathrm{var}^{-1}(\theta \mid P_1^{**})$ and $\tau_{\eta 2}$ has the U-shape depicted in Figure 1.

In the absence of contemporaneous public information, the effect of Y_2 on price informativeness is mainly determined by its effect on c_{12}/a_1. This fraction measures how strongly agents trade on their public non-fundamental information relative to their private fundamental information in the first period. How much weight traders put on Y_2 in the first period, in return, is shaped by two counteracting effects. As Y_2 predicts future noise trader demand more precisely, agents tend to put more weight on it when they form their demand. By contrast, a more precise Y_2 also implies that traders in the second period offset more of the influence of the second-period noise trader demand, as they trade against the signal. This

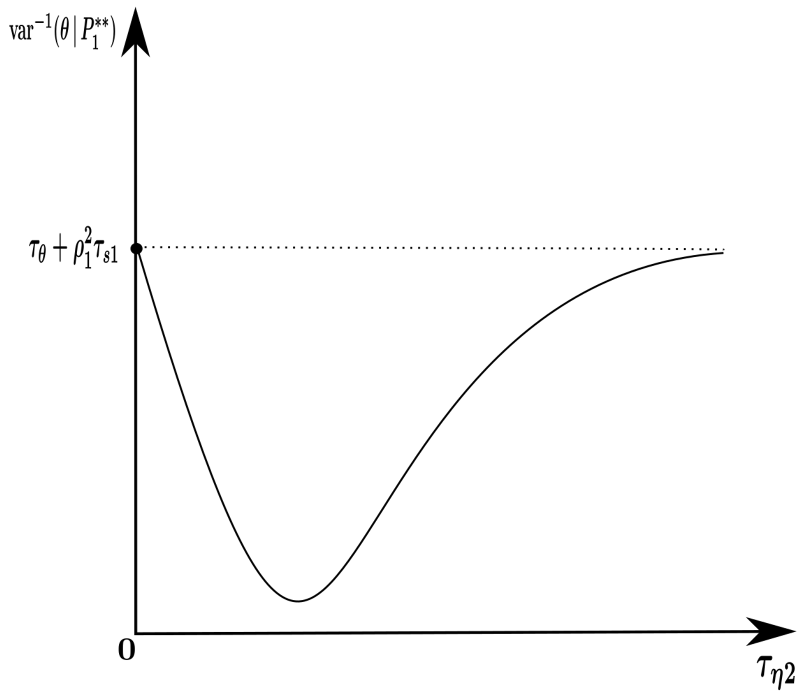

Figure 1: The impact of the precision of public information about future noise trader demand on current price informativeness

reduces the noise in the second-period price, which makes Y_2 less useful for predicting P_2 in the first period.

When public non-fundamental information is imprecise, the destabilizing effect dominates and agents trade more aggressively on Y_2 as it becomes more precise. However, there exists a point where the stabilizing effect takes over and agents in the first period trade less aggressively on Y_2 as it further gains in precision. In the limit, if information is perfectly precise, traders in the second period offset all noise inherent in the price. As a consequence, Y_2 is useless for predicting next period's price. In this situation, P_1 is not shaped by Y_2 and is, thus, as informative as when forward-looking information is completely imprecise.

In the case covered by Proposition 4 the informativeness of prices is a decreasing function of the precision of the public signal about noise trader demand for low levels of precision (as illustrated by Figure 1). The next proposition states the condition under which this is true in general (i.e., for $\tau_{\eta 1} \geq 0$).

Proposition 6: $\partial[\mathrm{var}^{-1}(\theta \mid P_1^{**})]/\partial\tau_{\eta 2} < 0$ *for* $\tau_{\eta 2} = 0$ *exactly if*

$$\frac{2\tau_{\eta 1}\tau_\varepsilon^2(1-\gamma\rho_2\tau_{s2})}{\tau_{s1}\left[\tau_\varepsilon + \rho_1^2(\tau_{s1}+\tau_{\eta 1})(1+\phi_1)\right]} < \frac{\tau_\theta + \rho_1^2(\tau_{s1}+\tau_{\eta 1})}{1+\gamma\rho_2\tau_{s2}},$$

where

$$\phi_1 = \frac{\gamma^2\left[\tau_\theta + \rho_1^2(\tau_{s1}+\tau_{\eta 1})\right]\tau_{s2}}{(1+\gamma\rho_2\tau_{s2})^2}.$$

The validity of the inequalities for $\tau_{\eta 1} = 0$ (the case treated in Proposition 5) is obvious. In the presence of contemporaneous public information, Y_2 exerts an additional effect by influencing how aggressively agents trade on Y_1 in the first period. The sign of $(1-\gamma\rho_2\tau_{s2})$ on the left-hand side of proposition 6 pins down whether agents trade more or less aggressively on Y_1 as the precision of Y_2 increases at $\tau_{\eta 2} = 0$. In both scenarios, the effect on the information role of Y_1 is more pronounced than the effect on its commonality role. If agents trade more aggressively on Y_1 (i.e., $1-\gamma\rho_2\tau_{s2} > 0$), the stabilizing effect of the information role dominates and the existence of Y_1 tends to stabilize prices. In this situation, it is ambiguous whether the existence of Y_2 harms price informativeness for sufficiently small values of $\tau_{\eta 2}$.
If agents trade less aggressively on Y_1 (i.e., $1-\gamma\rho_2\tau_{s2} < 0$), the reduction in its information role dominates the shrink in its commonality role and the existence of Y_1 tends to destabilize prices. In this situation, Y_2 unambiguously biases prices away from fundamentals.

4 Short-term trading and the Keynesian beauty contest

The model of Section 3 has a Keynesian beauty contest component (see Allen et al., 2006): the expected profitability of date-1 investments in the risky asset is determined by the expected date-2 price, which depends on average date-2 expectations about fundamentals, as revealed at date 3. This section shows that the main result, that higher precision of a public signal about noise trader demand may reduce price informativeness, also holds in a variant of the model with short-lived agents, in which the profitability of the first generation's investments depends *solely* on the date-2 resale price.

4.1 Model

The model is the same as in Section 3 except that there are two generations of rational traders, one that enters the market at date 1 and leaves at date 2 and another one that enters at date 2 and lives till date 3. Each set of investors has unit mass, and each investor is characterized by the same constant degree of risk tolerance γ. Each rational agent i obtains a private signal $x_i = \theta + \varepsilon_i$ about θ. For the sake of simplicity, there is no signal about date-1 noise trader demand (or $\tau_{\eta 1} = 0$). All rational traders observe $Y_2 = s_2 + \eta_2$. The exogenous random variables are jointly normally and independently distributed with the established notation for means and precisions. For simplicity, it is assumed that the variances of the noise trader shocks at the two dates are identical: $\tau_{s1} = \tau_{s2} \equiv \tau_s$.

4.2 Equilibrium

The final wealth of a first generation investor who invests D_{i1} in the risky asset $\pi_{i1} = (P_2 - P_1)D_{i1}$ is determined by the resale price P_2 and does not depend directly on asset fundamentals θ. Investors who enter the market at date 2 invest D_{i2} and obtain final wealth $\pi_{i2} = (\theta - P_2)D_{i2}$. An REE with short-lived agents is defined as a dynamic REE except that both generations of rational agents maximize their respective expected utility. Let the asset price be given by (10) and (11) with $c_{11} = c_{21} = 0$, as there is no signal about date-1 noise trader demand. Solving the model backwards yields a unique linear equilibrium (see the Appendix):

Proposition 7: *There exists a unique linear REE with short-lived agents, with*

$$\begin{aligned}
\rho_1 &= \frac{\tau_\varepsilon^2 \gamma^3 (\tau_s + \tau_{\eta 2})}{1 + \tau_\varepsilon \gamma^2 (\tau_s + \tau_{\eta 2})} \\
\rho_2 &= \gamma \tau_\varepsilon \\
\Lambda &= \tau_\theta + \tau_\varepsilon + \rho_1^2 \tau_s + \rho_2^2 (\tau_s + \tau_{\eta 2}) \\
a_1 &= \frac{\rho_1^2 \tau_s (\Lambda + \tau_\varepsilon) + \tau_\varepsilon^2 \left[1 + \gamma \rho_2 (\tau_s + \tau_{\eta 2})\right]}{\Lambda \left[\Lambda - \rho_2^2 (\tau_s + \tau_{\eta 2})\right]} \\
b_1 &= \frac{a_1}{\rho_1} \\
c_{12} &= \frac{\tau_{\eta 2}}{\gamma (\tau_s + \tau_{\eta 2}) \Lambda} \\
a_2 &= \frac{\tau_\varepsilon + \rho_2^2 (\tau_s + \tau_{\eta 2})}{\Lambda}
\end{aligned}$$

$$b_2 = \frac{1 + \gamma\rho_2(\tau_s + \tau_{\eta 2})}{\gamma\Lambda}$$

$$c_{22} = \frac{\rho_2\tau_{\eta 2} + \rho_1^2\tau_s\frac{c_{12}}{a_1}}{\Lambda}$$

$$d_2 = \frac{\rho_1^2\tau_s}{a_1\Lambda}.$$

Note that $\partial\rho_1/\partial\tau_{\eta 2} > 0$, i.e., a more precise signal about s_2 makes the first-period agents trade more aggressively on their fundamental information. The explanation for this is the following: As Y_2 reveals the second-period noise trader demand more precisely, the second-period agents face less uncertainty about the fundamental asset value, since they can infer it more accurately from disentangling P_2. This makes their expectations come closer to the actual value of θ, thereby weakening the impact of the Keynesian beauty contest. Furthermore, the second generation absorbs more of the second-period noise trader demand in this case, which mitigates its impact on P_2. That is, forecasting P_2 comes closer to forecasting θ as Y_2 becomes more precise. This allows the first-period agents to trade more aggressively on their fundamental information.

4.3 Price informativeness

Similar to the LLA model, we obtain

$$\text{var}^{-1}(\theta \mid P_1^{**}) = \tau_\theta + \left[\frac{1}{\rho_1^2\tau_s} + \left(\frac{c_{12}}{a_1}\right)^2\left(\frac{1}{\tau_s} + \frac{1}{\tau_{\eta 2}}\right)\right]^{-1}, \tag{16}$$

The effect of the introduction of Y_2 on the informativeness of P_1 is two-edged. On the one hand, it helps to weaken the Keynesian beauty contest. This allows the first generation to trade more aggressively on their fundamental signal, i.e., ρ_1 rises. This leads to more fundamental information being factored into the market price, thereby increasing its informativeness. On the other hand, it introduces an additional noisy component into the market price. This tends to decreases its informativeness. Comparative statics analysis of (16) with respect to $\tau_{\eta 2}$ immediately yields the next proposition:

Proposition 8: $\partial[\text{var}^{-1}(\theta \mid P_1^{**})]/\partial\tau_{\eta 2} < 0$ *for* $\tau_{\eta 2} = 0$ *exactly if*

$$\frac{2}{\rho_1\gamma\tau_\epsilon^2\tau_s} < C^2,$$

where

$$C = \frac{\tau_\theta + \tau_\epsilon + \rho_1^2 \tau_s}{\tau_\epsilon(1 + \gamma\rho_2\tau_s)(\tau_\epsilon + \rho_1\tau_s) + \rho_1^2\tau_s(\tau_\theta + \tau_\epsilon + \rho_1^2\tau_s)}.$$

The condition in proposition 8 can be further solved for a unique τ_θ. In the SLA model, forward-looking public non-fundamental information can also bias prices away from fundamentals although it weakens the Keynesian beauty contest.

5 Conclusion

This paper develops a clear contradiction to the traditional view gained from the competitive noisy rational expectations framework that all types of information boost price efficiency compared to the situation where they are absent. In our setup, public forward-looking non-fundamental information can harm the efficiency of prices. In the LLA model, if contemporaneous public information is absent, forward-looking information unambiguously lowers the efficiency of prices. This kind of information injects additional noise into the price, that is due to front-running noise trader demand. This extra noise drives prices away from fundamentals. If contemporaneous public non-fundamental information is present, forward-looking information can still be detrimental. In the SLA model, public forward-looking information exerts an additional stabilizing effect by weakening the Keynesian beauty contest. Nevertheless, it can harm price efficiency.

Our paper gives first insights into how specific types of non-fundamental information can bias prices away from fundamentals in a competitive trading environment with frictions. Nevertheless, there is still a lot of research to be done in this area. One next, promising route could be to extend our setup to the infinite-horizon case although this might be mathematically challenging. Additionally, one could add public *fundamental* information to the framework and explore the interactions of the different types of public information, which could potentially yield new results on market efficiency. These and other issues are left for future research.

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A Proofs

Proof of Proposition 3:

$$\mathrm{E}(\theta \,|\, I_{i2}) = \frac{\tau_\epsilon\, x_i + \rho_1^2\,(\tau_{s1} + \tau_{\eta1})\, P_1^* + \rho_2^2\,(\tau_{s2} + \tau_{\eta2})\, P_2^*}{\tau_\theta + \tau_\epsilon + \rho_1^2\,(\tau_{s1} + \tau_{\eta1}) + \rho_2^2\,(\tau_{s2} + \tau_{\eta2})}.$$

$$\mathrm{var}^{-1}(\theta \,|\, I_{i2}) = \tau_\theta + \tau_\epsilon + \rho_1^2\,(\tau_{s1} + \tau_{\eta1}) + \rho_2^2\,(\tau_{s2} + \tau_{\eta2})\,.$$

$$\frac{D_{2i}}{\gamma} = \tau_\epsilon\, x_i + \rho_1^2\,(\tau_{s1} + \tau_{\eta1})\, P_1^* + \rho_2^2\,(\tau_{s2} + \tau_{\eta2})\, P_2^* - \left[\tau_\theta + \tau_\epsilon + \rho_1^2\,(\tau_{s1} + \tau_{\eta1}) + \rho_2^2\,(\tau_{s2} + \tau_{\eta2})\right] P_2.$$

$$\begin{aligned}
0 &= \frac{s_2}{\gamma} + \int_0^1 \frac{D_{2i}}{\gamma} di \\
&= \frac{s_2}{\gamma} + \tau_\epsilon\, \theta + \rho_1^2\,(\tau_{s1} + \tau_{\eta1})\, P_1^* + \rho_2^2\,(\tau_{s2} + \tau_{\eta2})\, P_2^* \\
&\quad - \left[\tau_\theta + \tau_\epsilon + \rho_1^2\,(\tau_{s1} + \tau_{\eta1}) + \rho_2^2\,(\tau_{s2} + \tau_{\eta2})\right] P_2 \\
&= \frac{s_2}{\gamma} + \tau_\epsilon\, \theta + \rho_1^2\,(\tau_{s1} + \tau_{\eta1}) \left(\frac{P_1 + c_{11} Y_1 - c_{12} Y_2}{a_1} - \frac{1}{\rho_1} \frac{\tau_{\eta1}}{\tau_{s1} + \tau_{\eta1}} Y_1 \right) \\
&\quad + \rho_2^2\,(\tau_{s2} + \tau_{\eta2}) \left[\theta + \frac{1}{\rho_2} \left(s_2 - \frac{\tau_{\eta2}}{\tau_{s2} + \tau_{\eta2}} Y_2 \right) \right] \\
&\quad - \left[\tau_\theta + \tau_\epsilon + \rho_1^2\,(\tau_{s1} + \tau_{\eta1}) + \rho_2^2\,(\tau_{s2} + \tau_{\eta2})\right] P_2.
\end{aligned}$$

$$\begin{aligned}
P_2 = \frac{1}{\Delta} \Bigg\{ & \left[\tau_\epsilon + \rho_2^2\,(\tau_{s2} + \tau_{\eta2})\right] \theta + \left[\gamma^{-1} + \rho_2^2\,(\tau_{s2} + \tau_{\eta2})\right] s_2 \\
& - \left[\rho_1 \tau_{\eta1} - \rho_1^2\,(\tau_{s1} + \tau_{\eta1}) \frac{c_{11}}{a_1} \right] Y_1 - \left[\rho_2 \tau_{\eta2} + \rho_1^2\,(\tau_{s1} + \tau_{\eta1}) \frac{c_{12}}{a_1} \right] Y_2 \\
& + \rho_1^2\,(\tau_{s_1} + \tau_{\eta1}) \frac{1}{a_1} P_1 \Bigg\}
\end{aligned}$$

$$a_2 = \frac{\tau_\epsilon + \rho_2^2\,(\tau_{s2} + \tau_{\eta2})}{\Delta}$$

$$b_2 = \frac{1 + \gamma \rho_2\,(\tau_{s2} + \tau_{\eta2})}{\gamma \Delta}$$

$$\rho_2 = \gamma \tau_\epsilon$$

$$c_{21} = \frac{a_1 \rho_1 \tau_{\eta 1} - \rho_1^2 \left(\tau_{s1} + \tau_{\eta 1}\right) c_{11}}{a_1 \Delta}$$

$$c_{22} = \frac{a_1 \rho_2 \tau_{\eta 2} + \rho_1^2 \left(\tau_{s1} + \tau_{\eta 1}\right) c_{12}}{a_1 \Delta}$$

$$d_2 = \frac{\rho_1^2 \left(\tau_{s1} + \tau_{\eta 1}\right)}{a_1 \Delta}$$

Denote $\Gamma_1 \equiv \mathrm{var}(\theta \,|\, I_{i1})$ and $\Gamma_2 = \mathrm{var}(s_2 \,|\, I_{i1})$

$$\mathrm{cov}(\theta - P_2, P_2 \,|\, I_{i1}) = a_2(1 - a_2)\Gamma_1 - b_2^2 \Gamma_2$$

$$\mathrm{var}(\theta - P_2 \,|\, I_{i1}) = (1 - a_2)^2 \Gamma_1 + b_2^2 \Gamma_2$$

$$h = \frac{a_2(1 - a_2)\Gamma_1 - b_2^2 \Gamma_2}{(1 - a_2)^2 \Gamma_1 + b_2^2 \Gamma_2}$$

$$\begin{aligned} \mathrm{E}[P_2 - h(\theta - P_2) \,|\, I_{i1}] = {} & [(1 + h)a_2 - h]\, \mathrm{E}(\theta \,|\, I_{i1}) \\ & + (1 + h) \left[b_2 \mathrm{E}(s_2 \,|\, I_{i1}) - c_{21} Y_1 - c_{22} Y_2 + d_2 P_1\right] \end{aligned}$$

$$\mathrm{E}(\theta \,|\, I_{i1}) = \frac{\tau_\epsilon \, x_i + \rho_1^2 \left(\tau_{s1} + \tau_{\eta 1}\right) P_1^*}{\tau_\theta + \tau_\epsilon + \rho_1^2 \left(\tau_{s1} + \tau_{\eta 1}\right)}$$

$$\mathrm{E}(s_2 \,|\, I_{i1}) = \frac{\tau_{\eta 2}}{\tau_{s2} + \tau_{\eta 2}} Y_2$$

$$\mathrm{var}[P_2 - h(\theta - P_2) \,|\, I_{i1}] = \frac{b_2^2 \Gamma_1 \Gamma_2}{(1 - a_2)^2 \Gamma_1 + b_2^2 \Gamma_2}$$

$$\begin{aligned} \frac{D_{1i}}{\gamma} = {} & \tau_\epsilon \, x_i + \rho_1^2 \left(\tau_{s1} + \tau_{\eta 1}\right) P_1^* + \frac{1 - a_2}{b_2^2 \Gamma_2} \left(b_2 \frac{\tau_{\eta 2}}{\tau_{s2} + \tau_{\eta 2}} Y_2 - c_{21} Y_1 - c_{22} Y_2 + d_2 P_1 \right) \\ & - \frac{(1 - a_2)^2 \Gamma_1 + b_2^2 \Gamma_2}{b_2^2 \Gamma_1 \Gamma_2} P_1 \end{aligned}$$

$$
\begin{aligned}
0 &= \frac{s_1}{\gamma} + \int_0^1 \frac{D_{1i}}{\gamma} di \\
&= \frac{s_1}{\gamma} + \tau_\epsilon \, \theta + \rho_1^2 \left(\tau_{s1} + \tau_{\eta 1}\right) P_1^* + \frac{1-a_2}{b_2^2 \Gamma_2} \left(b_2 \frac{\tau_{\eta 2}}{\tau_{s2} + \tau_{\eta 2}} Y_2 - c_{21} Y_1 - c_{22} Y_2 + d_2 P_1 \right) \\
&\quad - \frac{(1-a_2)^2 \Gamma_1 + b_2^2 \Gamma_2}{b_2^2 \Gamma_1 \Gamma_2} P_1 \\
&= \tau_\epsilon \, \theta + \frac{s_1}{\gamma} - \left[\rho_1 \tau_{\eta 1} - \rho_1^2 \left(\tau_{s_1} + \tau_{\eta 1}\right) \frac{c_{11}}{a_1} + \frac{1-a_2}{b_2^2 \Gamma_2} \, c_{21} \right] Y_1 \\
&\quad + \left[\frac{1-a_2}{b_2^2 \Gamma_2} \left(b_2 \frac{\tau_{\eta 2}}{\tau_{s2} + \tau_{\eta 2}} - c_{22} \right) - \rho_1^2 \left(\tau_{s1} + \tau_{\eta 1}\right) \frac{c_{12}}{a_1} \right] Y_2 \\
&\quad - \left[\frac{(1-a_2)^2 \Gamma_1 + b_2^2 \Gamma_2}{b_2^2 \Gamma_1 \Gamma_2} - \frac{1-a_2}{b_2^2 \Gamma_2} \, d_2 - \rho_1^2 \left(\tau_{s1} + \tau_{\eta 1}\right) \frac{1}{a_1} \right] P_1 .
\end{aligned}
$$

$$
\rho_1 = \gamma \tau_\epsilon
$$

$$
\frac{\tau_\epsilon}{a_1} = \frac{(1-a_2)^2 \Gamma_1 + b_2^2 \Gamma_2}{b_2^2 \Gamma_1 \Gamma_2} - \frac{1-a_2}{b_2^2 \Gamma_2} \frac{\rho_1^2 \left(\tau_{s1} + \tau_{\eta 1}\right)}{a_1 \Delta} - \rho_1^2 \left(\tau_{s1} + \tau_{\eta 1}\right) \frac{1}{a_1}
$$

$$
\Leftrightarrow a_1 = \frac{\left[\left(\tau_\epsilon + \rho_1^2 \left(\tau_{s1} + \tau_{\eta 1}\right)\right] b_2^2 \Gamma_1 \Gamma_2}{(1-a_2)^2 \Gamma_1 + b_2^2 \Gamma 2} + \frac{(1-a_2) \Gamma_1}{(1-a_2)^2 \Gamma_1 + b_2^2 \Gamma_2} \frac{\rho_1^2 (\tau_{s1} + \tau_{\eta 1})}{\Delta}
$$

$$
b_1 = \frac{a_1}{\rho_1}
$$

$$
\begin{aligned}
\frac{c_{11}}{a_1} \; &= \frac{\rho_1 \tau_{\eta 1} - \rho_1^2 \left(\tau_{s_1} + \tau_{\eta 1}\right) \frac{c_{11}}{a_1} + \frac{1-a_2}{b_2^2 \Gamma_2} \, c_{21}}{\tau_\epsilon} \\
&= \frac{\rho_1 \tau_{\eta 1} - \rho_1^2 \left(\tau_{s_1} + \tau_{\eta 1}\right) \frac{c_{11}}{a_1} + \frac{1-a_2}{b_2^2 \Gamma_2} \frac{\rho_1 \tau_{\eta 1} - \rho_1^2 \left(\tau_{s1} + \tau_{\eta 1}\right) \frac{c_{11}}{a_1}}{\Delta}}{\tau_\epsilon} \\
\Leftrightarrow \frac{c_{11}}{a_1} &= \frac{\rho_1 \tau_{\eta 1} \left(1 + \frac{1-a_2}{\Delta b_2^2 \Gamma_2} \right)}{\tau_\epsilon + \rho_1^2 \left(\tau_{s1} + \tau_{\eta 1}\right) \left(1 + \frac{1-a_2}{\Delta b_2^2 \Gamma_2} \right)}
\end{aligned}
$$

$$\frac{c_{12}}{a_1} = \frac{\frac{1-a_2}{b_2^2\Gamma_2}\left(b_2\frac{\tau_{\eta 2}}{\tau_{s2}+\tau_{\eta 2}} - c_{22}\right) - \rho_1^2\left(\tau_{s1}+\tau_{\eta 1}\right)\frac{c_{12}}{a_1}}{\tau_\epsilon}$$

$$= \frac{\frac{1-a_2}{b_2^2\Gamma_2}\left(b_2\frac{\tau_{\eta 2}}{\tau_{s2}+\tau_{\eta 2}} - \frac{\rho_2\tau_{\eta 2}+\rho_1^2\left(\tau_{s1}+\tau_{\eta 1}\right)\frac{c_{12}}{a_1}}{\Delta}\right) - \rho_1^2\left(\tau_{s1}+\tau_{\eta 1}\right)\frac{c_{12}}{a_1}}{\tau_\epsilon}$$

$$\Leftrightarrow \frac{c_{12}}{a_1} = \frac{\frac{1-a_2}{b_2}\tau_{\eta 2}\left(1-\frac{\rho_2}{\Delta b_2\Gamma_2}\right)}{\tau_\epsilon + \rho_1^2\left(\tau_{s1}+\tau_{\eta 1}\right)\left(1+\frac{1-a_2}{\Delta b_2^2\Gamma_2}\right)}. \quad \text{q.e.d.}$$

Proof of Proposition 4: Define

$$\begin{aligned}
A_1 &\equiv \left(\frac{1}{\rho_1}-\frac{c_{11}}{a_1}\right)^2\frac{1}{\tau_{s1}} \\
A_2 &\equiv +\left(\frac{c_{11}}{a_1}\right)^2\frac{1}{\tau_{\eta 1}} \\
A_3 &\equiv \left(\frac{c_{12}}{a_1}\right)^2\left(\frac{1}{\tau_{s2}}+\frac{1}{\tau_{\eta 2}}\right),
\end{aligned}$$

so that $\mathrm{var}^{-1}(\theta \mid P_1^{**}) = \tau_\theta + (A_1+A_2+A_3)^{-1}$. From Proposition 3,

$$\begin{aligned}
\frac{1-a_2}{b_2} &\to 0 \\
\frac{1-a_2}{b_2}\tau_{\eta 2} &\to \frac{\tau_\theta+\rho_1^2(\tau_{s1}+\tau_{\eta 1})}{\rho_2} \\
b_2\Gamma_2\Delta &\to \rho_2 \\
\frac{c_{11}}{a_1} &\to \frac{\rho_1\tau_{\eta 1}}{\tau_\varepsilon+\rho_1^2(\tau_{s1}+\tau_{\eta 1})} \\
\frac{c_{12}}{a_1} &\to 0 \\
A_1 &\to \frac{1}{\rho_1^2\tau_{s1}}\left[\frac{\tau_\varepsilon+\rho_1^2\tau_{s1}}{\tau_\varepsilon+\rho_1^2(\tau_{s1}+\tau_{\eta 1})}\right]^2 \\
A_2 &\to \frac{1}{\rho_1^2\tau_{\eta 1}}\left[\frac{\rho_1^2\tau_{\eta 1}}{\tau_\varepsilon+\rho_1^2(\tau_{s1}+\tau_{\eta 1})}\right]^2 \\
A_3 &\to 0
\end{aligned}$$

as $\tau_{\eta 2} \to \infty$ and

$$\begin{aligned}
\frac{1-a_2}{b_2} &\to \frac{\tau_\theta + \rho_1^2(\tau_{s1}+\tau_{\eta 1})}{1+\gamma\rho_2\tau_{s2}} \\
b_2\Gamma_2\Delta &\to \frac{1+\gamma\rho_2\tau_{s2}}{\gamma\tau_{s2}} \\
\frac{c_{11}}{a_1} &\to \frac{\rho_1\tau_{\eta 1}}{B\tau_\varepsilon + \rho_1^2(\tau_{s1}+\tau_{\eta 1})} \\
\frac{c_{12}}{a_1} &\to 0 \\
A_1 &\to \frac{1}{\rho_1^2\tau_{s1}}\left[\frac{B\tau_\varepsilon + \rho_1^2\tau_{s1}}{B\tau_\varepsilon + \rho_1^2(\tau_{s1}+\tau_{\eta 1})}\right]^2 \\
A_2 &\to \frac{1}{\rho_1^2\tau_{\eta 1}}\left[\frac{\rho_1^2\tau_{\eta 1}}{B\tau_\varepsilon + \rho_1^2(\tau_{s1}+\tau_{\eta 1})}\right]^2 \\
A_3 &\to 0
\end{aligned}$$

as $\tau_{\eta 2} \to 0$, where

$$B \equiv \left[1 + \gamma\tau_{s2}\frac{\tau_\theta + \rho_1^2(\tau_{s1}+\tau_{\eta 1})}{(1+\gamma\rho_2\tau_{s2})^2}\right]^{-1} < 1.$$

It is easily checked that $A_1 + A_2 + A_3$ is smaller for $\tau_{\eta 2} \to 0$ than for $\tau_{\eta 2} \to \infty$. q.e.d.

Proof of Proposition 5: The results stated in the main text follow from inspection of the expressions in Proposition 3.

$$\begin{aligned}
\frac{1-a_2}{b_2} &= \gamma\frac{\tau_\theta + \rho_1^2\tau_{s1}}{1+\gamma\rho_2(\tau_{s2}+\tau_{\eta 2})} \\
b_2\Gamma_2\Delta &= \frac{1+\gamma\rho_2(\tau_{s2}+\tau_{\eta 2})}{\gamma(\tau_{s2}+\tau_{\eta 2})}
\end{aligned}$$

are independent of $\tau_{\eta 1}$. It follows that

$$\begin{aligned}
\frac{c_{11}}{a_1} &= 0 \\
\left(\frac{c_{11}}{a_1}\right)^2\frac{1}{\tau_{\eta 1}} &= 0
\end{aligned} \tag{A.1}$$

for $\tau_{\eta_1} = 0$. That $c_{12}/a_1 = 0$ only if $\tau_{\eta 2} = 0$ follows from the fact that

$$\frac{\rho_2}{b_2\Gamma_2\Delta} = \frac{\gamma\rho_2(\tau_{s2}+\tau_{\eta 2})}{1+\gamma\rho_2(\tau_{s2}+\tau_{\eta 2})} < 1.$$

q.e.d.

Proof of Proposition 7:

The vectors of signals available to rational agents born at dates 1 and 2 are $I_{i1} = (P_1, x_i, Y_2)$

and $I_{i2} = (P_1, x_i, Y_2, P_2)$, respectively. Define P_1^* and P_2^* as in Section 3, with $c_{11} = c_{21} = 0$, $\mathrm{E}(s_1 \,|\, Y_1) = 0$, and $\tau_{\eta 1} = 0$.

The problem of using Y_2 in order to update expectations about θ is exactly the same as in Section 3: P_2^* is a signal about θ with precision $\rho_2^2(\tau_{s2} + \tau_{\eta 2})$. As such, it helps to predict the asset's payoff and, hence, the date-2 demand for and price of the asset.

Date 2:

$$\mathrm{E}(\theta \,|\, I_{2i}) = \frac{\rho_1^2 \tau_s P_1^* + \tau_\varepsilon x_i + \rho_2^2(\tau_s + \tau_{\eta 2}) P_2^*}{\tau_\theta + \tau_\varepsilon + \rho_1^2 \tau_s + \rho_2^2(\tau_s + \tau_{\eta 2})}$$

$$\mathrm{var}(\theta \,|\, I_{2i}) = \frac{1}{\tau_\theta + \tau_\varepsilon + \rho_1^2 \tau_s + \rho_2^2(\tau_s + \tau_{\eta 2})}.$$

$$\begin{aligned}
D_{2i} &= \gamma \frac{\mathrm{E}(\theta \,|\, I_{2i}) - P_2}{\mathrm{var}(\theta \,|\, I_{2i})} \\
&= \gamma \frac{\dfrac{\rho_1^2 \tau_s P_1^* + \tau_\varepsilon x_i + \rho_2^2(\tau_s + \tau_{\eta 2}) P_2^*}{\tau_\theta + \tau_\varepsilon + \rho_1^2 \tau_s + \rho_2^2(\tau_s + \tau_{\eta 2})} - P_2}{[\tau_\theta + \tau_\varepsilon + \rho_1^2 \tau_s + \rho_2^2(\tau_s + \tau_{\eta 2})]^{-1}} \\
&= \gamma \left[\frac{\rho_1^2 \tau_s}{a_1} P_1 + \tau_\varepsilon x_i + \rho_2^2(\tau_s + \tau_{\eta 2})\theta - \left(\rho_1^2 \tau_s \frac{c_{12}}{a_1} + \rho_2 \tau_{\eta 2} \right) Y_2 \right. \\
&\quad \left. + \rho_2(\tau_s + \tau_{\eta 2}) s_2 - P_2(\tau_\theta + \tau_\varepsilon + \rho_1^2 \tau_s + \rho_2^2(\tau_s + \tau_{\eta 2}) \right].
\end{aligned}$$

From market clearing at date 2 ($\int_0^1 D_{2i} di + s_2 = 0$) and the strong law of large numbers ($\int_0^1 x_{2i} di = \theta$),

$$\begin{aligned}
0 &= \gamma \left\{ \frac{\rho_1^2 \tau_s}{a_1} P_1 + \tau_\varepsilon \theta + \rho_2^2(\tau_s + \tau_{\eta 2})\theta - \left(\rho_1^2 \tau_s \frac{c_{12}}{a_1} + \rho_2 \tau_{\eta 2} \right) Y_2 \right. \\
&\quad \left. + \rho_2(\tau_s + \tau_{\eta 2}) s_2 - \left[\tau_\theta + \tau_\varepsilon + \rho_1^2 \tau_s + \rho_2^2(\tau_s + \tau_{\eta 2}) \right] P_2 \right\} + s_2 \\
\Leftrightarrow P_2 &= \frac{\tau_\varepsilon + \rho_2^2(\tau_s + \tau_{\eta 2})}{\tau_\theta + \tau_\varepsilon + \rho_1^2 \tau_s + \rho_2^2(\tau_s + \tau_{\eta 2})} \theta + \frac{1 + \gamma \rho_2(\tau_s + \tau_{\eta 2})}{\gamma \left[\tau_\theta + \tau_\varepsilon + \rho_1^2 \tau_s + \rho_2^2(\tau_s + \tau_{\eta 2}) \right]} s_2 \\
&\quad - \frac{\rho_1^2 c_{12} \tau_s + a_1 \rho_2 \tau_{\eta 2}}{a_1 \left[\tau_\theta + \tau_\varepsilon + \rho_1^2 \tau_s + \rho_2^2(\tau_s + \tau_{\eta 2}) \right]} Y_2 + \frac{\rho_1^2 \tau_s}{a_1 \left[\tau_\theta + \tau_\varepsilon + \rho_1^2 \tau_s + \rho_2^2(\tau_s + \tau_{\eta 2}) \right]} P_1.
\end{aligned}$$

Equating the coefficients to those in (11) yields the expressions for a_2, b_2, c_{22}, and d_2 in Proposition 7.

Date 1:

$$\mathrm{E}(\theta \,|\, I_{1i}) = \frac{\rho_1^2 \tau_s P_1^* + \tau_\varepsilon x_i}{\tau_\theta + \tau_\varepsilon + \rho_1^2 \tau_s}$$

$$\mathrm{var}(\theta \,|\, I_{1i}) = \frac{1}{\tau_\theta + \tau_\varepsilon + \rho_1^2 \tau_s}.$$

$$
\begin{aligned}
\mathrm{E}(P_2 \,|\, I_{1i}) &= \frac{\tau_\varepsilon + \rho_2^2(\tau_s + \tau_{\eta 2})}{\Lambda}\mathrm{E}(\theta \,|\, I_{1i}) + \frac{1 + \gamma\rho_2(\tau_s + \tau_{\eta 2})}{\gamma\Lambda}\mathrm{E}(s_2 \,|\, I_{1i}) \\
&\quad - \frac{\rho_1^2 c_{12}\tau_s + a_1\rho_2\tau_{\eta 2}}{a_1\Lambda}\mathrm{E}(Y_2 \,|\, I_{1i}) + \frac{\rho_1^2\tau_s}{a_1\Lambda}\mathrm{E}(P_1 \,|\, I_{1i}) \\
&= \frac{1}{\Lambda}\Bigg(\frac{\left[\tau_\varepsilon + \rho_2^2(\tau_s + \tau_{\eta 2})\right]\tau_\varepsilon}{\tau_\theta + \tau_\varepsilon + \rho_1^2\tau_s} x_{1i} + \left\{ \frac{\left[\tau_\varepsilon + \rho_2^2(\tau_s + \tau_{\eta 2})\right]\rho_1^2\tau_s}{a_1(\tau_\theta + \tau_\varepsilon + \rho_1^2\tau_s)} + \frac{\rho_1^2\tau_s}{a_1} \right\} P_1 \\
&\quad + \Bigg\{ - \frac{\left[\tau_\varepsilon + \rho_2^2(\tau_s + \tau_{\eta 2})\right]\rho_1^2\tau_s c_{12}}{a_1(\tau_\theta + \tau_\varepsilon + \rho_1^2\tau_s)} + \frac{\left[1 + \gamma\rho_2(\tau_s + \tau_{\eta 2})\right]\tau_{\eta 2}}{\gamma(\tau_s + \tau_{\eta 2})} \\
&\quad - \frac{a_1\rho_2\tau_{\eta 2} + \rho_1^2 c_{12}\tau_s}{a_1} \Bigg\} \Bigg) Y_2
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{var}(P_2 \,|\, I_{1i}) &= \left[\tau_\varepsilon + \rho_2^2(\tau_s + \tau_{\eta 2})\Lambda\right]^2 \mathrm{var}(\theta \,|\, I_{1i}) + \left[\frac{1 + \gamma\rho_2(\tau_s + \tau_{\eta 2})}{\gamma\Lambda}\right]^2 \mathrm{var}(s_2 \,|\, I_{1i}) \\
&= \left[\frac{\tau_\varepsilon + \rho_2^2(\tau_s + \tau_{\eta 2})}{\Lambda}\right]^2 \frac{1}{\tau_\theta + \tau_\varepsilon + \rho_1^2\tau_s} + \left[\frac{1 + \gamma\rho_2(\tau_s + \tau_{\eta 2})}{\gamma\Lambda}\right]^2 \frac{1}{\tau_s + \tau_{\eta 2}}
\end{aligned}
$$

Following the standard argument regarding the maximization of expected utility, the demand for the risky asset of an agent i in the first period is given by

$$
D_{1i} = \gamma \frac{\mathrm{E}\left(P_2 | I_{1i}\right) - P_1)}{\mathrm{Var}\left(P_2 | I_{1i}\right)}.
$$

Define

$$
\begin{aligned}
&\alpha \equiv \frac{(\tau_\varepsilon + \rho_2^2(\tau_s + \tau_{\eta 2}))\tau_\varepsilon}{\tau_\theta + \tau_\varepsilon + \rho_1^2\tau_s}, \;\; \beta \equiv \frac{(\tau_\varepsilon + \rho_2^2(\tau_s + \tau_{\eta 2}))\rho_1^2\tau_s}{a_1(\tau_\theta + \tau_\varepsilon + \rho_1^2\tau_s)} + \frac{\rho_1^2\tau_s}{a_1}, \quad \text{and} \\
&\delta \equiv -\frac{(\tau_\varepsilon + \rho_2^2(\tau_s + \tau_{\eta 2}))\rho_1^2\tau_s\, c_{12}}{a_1(\tau_\theta + \tau_\varepsilon + \rho_1^2\tau_s)} + \frac{(1 + \gamma\rho_2(\tau_s + \tau_{\eta 2}))\tau_{\eta 2}}{\gamma(\tau_s + \tau_{\eta 2})} - \frac{a_1\rho_2\tau_{\eta 2} + \rho_1^2 c_{12}\tau_s}{a_1}.
\end{aligned}
$$

Then, market clearing in the first period implies that

$$
\int_0^1 D_{1i}\,\mathrm{d}i + s_1 = \frac{\gamma}{\Lambda}\, \frac{\alpha\,\theta + (\beta - \Lambda)P_1 + \delta Y_2}{\dfrac{(\tau_\varepsilon + \rho_2^2(\tau_s + \tau_{\eta 2}))^2}{\Lambda^2(\tau_\theta + \tau_\varepsilon + \rho_1^2\tau_s)} + \dfrac{(1 + \gamma\rho_2(\tau_s + \tau_{\eta 2}))^2}{\gamma^2\Lambda^2(\tau_s + \tau_{\eta 2})}} + s_1 = 0.
$$

Solving for P_1 and matching coefficients yields the expression for ρ_1, a_1, b_1, and c_{12} in proposition 7. q.e.d.